

COMPLEX HYPERBOLIC HYPERPLANE COMPLEMENTS

IGOR BELEGRADEK

ABSTRACT. We study spaces obtained from a complete finite volume complex hyperbolic n -manifold M by removing a compact totally geodesic complex $(n-1)$ -submanifold S . The main result is that the fundamental group of $M \setminus S$ is relatively hyperbolic, relative to fundamental groups of the ends of $M \setminus S$, and $M \setminus S$ admits a complete finite volume A -regular Riemannian metric of negative sectional curvature.

It follows that for $n > 1$ the fundamental group of $M \setminus S$ satisfies Mostow-type Rigidity, has finite asymptotic dimension and rapid decay property, satisfies Borel and Baum-Connes conjectures, is co-Hopf and residually hyperbolic, has no nontrivial subgroups with property (T), and has finite outer automorphism group. Furthermore, if M is compact, then the fundamental group of $M \setminus S$ is biautomatic and satisfies Strong Tits Alternative.

1. INTRODUCTION

Let M be a (connected) complete finite volume complex hyperbolic n -manifold, and let S be a (possibly disconnected) compact totally geodesic complex submanifold of dimension $(n-1)$; so the pair (M, S) is modelled on $(\mathbf{CH}^n, \mathbf{CH}^{n-1})$ where \mathbf{CH}^n denotes the complex hyperbolic symmetric space of dimension n . This paper is a sequel to [Bel], where we studied $M \setminus S$ in the case when M, S are real hyperbolic and S has real codimension two. Clearly $M \setminus S$ can be identified with the interior of a compact smooth manifold N that is obtained from M by removing a tubular neighborhood of S and chopping off all cusps (in case M is noncompact). There are two kinds of components of ∂N : compact infranil manifolds appearing as cusp cross-sections of M , and circle bundles over components of S . The main technical result of this paper is

Theorem 1.1. *If M is a complete finite volume complex hyperbolic n -manifold, and S is a compact totally geodesic complex $(n-1)$ -submanifold, then*

- (i) *$M \setminus S$ admits a complete finite volume metric of $\sec \leq -1$;*
- (ii) *the group $\pi_1(N)$ is non-elementary (strongly) relatively hyperbolic, where the peripheral subgroups are fundamental groups of the components of ∂N .*
- (iii) *$M \setminus S$ admits a complete finite volume A -regular metric of $\sec < 0$.*

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The proof of Part (i) involves a delicate warped product computation which occupies most of this paper, and is sketched in Section 2. Once the metric is constructed, it follows from [Bel, Section 4] that $\pi_1(N)$ satisfies Gromov's definition of relative hyperbolicity elaborated in [Bow].

Recall that a Riemannian metric is called *A-regular* if there exists a sequence of positive numbers $A = \{A_k\}$ such that for each $k \geq 0$, the k th covariant derivative of the curvature tensor satisfies $\|\nabla^k R\|_{C^0} < A_k$. Any metric on a compact manifold is *A-regular*, and similarly, this is true e.g. for an open manifold which is isometric outside a compact subset to the product of a closed manifold and a ray. A locally symmetric metric is *A-regular* because the sectional curvature is bounded and the covariant derivative of the curvature tensor vanishes. A complete Riemannian metric with $a \leq \sec \leq b$ admits a C^1 -nearby complete *A-regular* metric with almost the same curvature bounds [Kap05], in particular, if a, b are negative, then the *A-regular* metric is negatively pinched; however, if $b = 0$, then the *A-regular* metric of [Kap05] need not be non-positively curved. In general, proving that a complete nonpositively curved manifold admits a complete *A-regular* metric nonpositively curved metric is quite difficult, and in Theorem 1.1(i) this is done by modifying the metric constructed in (i), and proving *A-regularity* by a brute force computation. The existence of *A-regular* nonpositively curved metrics has striking topological consequences, namely Farrell-Jones [FJ98, Addendum 0.5] proved that the fundamental group of any complete manifold with *A-regular* metric of nonpositive curvature satisfies Borel's conjecture, while Lafforgue [Laf02, Corollary 0.0.4] proved Baum-Connes conjecture for the fundamental groups of complete *A-regular* nonpositively curved manifolds that satisfy Rapid Decay property.

It is shown in [Bel, Section 6] that the following Mostow-type rigidity result is implied by part (ii) of Theorem 1.1 combined with the classical Mostow-Prasad Rigidity.

Theorem 1.2. *For $n > 1$ and $i = 1, 2$, suppose that M_i is a complete finite volume complex hyperbolic n -manifold, and S_i is a compact totally geodesic complex $(n - 1)$ -submanifold. Then any homotopy equivalence $f: M_1 \setminus S_1 \rightarrow M_2 \setminus S_2$ induces an isometry $\iota_f: M_1 \rightarrow M_2$ taking S_1 to S_2 such that the restriction $\iota_f: M_1 \setminus S_1 \rightarrow M_2 \setminus S_2$ is homotopic to f . Moreover, ι_f is uniquely determined by the homotopy class of f .*

Corollary 1.3. *For $n > 1$ if M is a complete finite volume complex hyperbolic n -manifold, and S is a compact totally geodesic complex $(n - 1)$ -submanifold, then the correspondence $f \rightarrow \iota_f$ induces an isomorphism of the outer automorphism group of $\pi_1(M \setminus S)$ onto the group of isometries of M that map S to itself. In particular, the outer automorphism group of $\pi_1(M \setminus S)$ is finite.*

Other corollaries of Theorem 1.1 are summarized below.

Theorem 1.4. *If M is a complete finite volume complex hyperbolic n -manifold, and S is a compact totally geodesic complex $(n - 1)$ -submanifold with $n > 1$, then*

- (1) *the relatively hyperbolic boundary of $\pi_1(N)$ is the $(n - 1)$ -sphere.*
- (2) *$\pi_1(N)$ does not split as an amalgamated product or an HNN-extension over subgroups of peripheral subgroups of $\pi_1(N)$, or over \mathbb{Z} .*
- (3) *$\pi_1(N)$ is co-Hopf.*
- (4) *for any finite subset $F \subset \pi_1(N)$ there is a homomorphism of $\pi_1(N)$ onto a non-elementary hyperbolic group that is injective on F .*
- (5) *$\pi_1(N)$ satisfies Strong Tits Alternative iff M is compact.*
- (6) *$\pi_1(N)$ is biautomatic iff M is compact.*
- (7) *No nontrivial subgroup of $\pi_1(N)$ has Kazhdan property (T).*
- (8) *$\pi_1(N)$ is not a CAT(0) group.*
- (9) *$\pi_1(N)$ has finite asymptotic dimension.*
- (10) *$\pi_1(N)$ has rapid decay property.*
- (11) *$\pi_1(N)$ satisfies Baum-Connes conjecture.*
- (12) *$\pi_1(N)$ satisfies Borel Conjecture, and in particular, if $n > 2$, then any homotopy equivalence of compact manifolds $L \rightarrow N$ that restricts to a homeomorphism of the boundaries $\partial L \rightarrow \partial N$ is homotopic to a homeomorphism rel boundary.*
- (13) *$\pi_1(N)$ is not isomorphic to the fundamental group of a complete negatively pinched Riemannian manifold.*
- (14) *if $\pi_1(N)$ is isomorphic to a lattice Λ in a real Lie group G , then the identity component G_0 of G is compact, $\Lambda \cap G_0$ is trivial, and Λ projects isomorphically onto a finite index subgroup of G/G_0 .*

Theorems 1.2–1.4 and Corollary 1.3 easily follow by combining Theorem 1.1 with various (often deep) works available in the literature, and with a few exceptions, their proofs are identical to the proofs of the corresponding results in the real hyperbolic case written in [Bel]; the cases where the proofs are different from those in [Bel] are dealt with in Section 13.

Before this paper, there has been almost no research done on the topology and geometry of $M \setminus S$, and the only exceptions known to me are as follows. Arguments of Toledo imply that the group $\pi_1(M \setminus S)$ is not Kähler when $n = 2$ [ABC⁺96, page 112], and is sometimes Kähler when $n > 2$ [Tol93, pages 107–110] (neither of these references treats the case of $M \setminus S$ directly, but the proofs still work with minor modifications). Allcock-Carlson-Toledo [ACT02] studied a more general (and much more complicated) case when hyperplanes in S are allowed to intersect orthogonally; they write down an explicit infinite presentation for the kernel of the homomorphism $\pi_1(M \setminus S) \rightarrow \pi_1(M)$ induced by the inclusion, and prove that $\pi_1(M \setminus S)$ is not isomorphic to a lattice in a virtually connected real Lie group.

I refer to [Bel] for some open problems about $M \setminus S$, and limit myself to discussing the following tantalizing question due to Toledo.

Question 1.5. *Is $\pi_1(M \setminus S)$ residually finite?*

Toledo [Tol93] showed that the answer is generally *no* in the similar case when (M, S) is modelled on $(\mathbf{X}_n, \mathbf{X}_{n-1})$ where \mathbf{X}_n is the symmetric space for $SO(n, 2)$. It is instructive to recall his argument. Fix an arbitrary component B of the boundary of a small tubular neighborhood of S in M . Toledo proves that the inclusion $B \rightarrow M \setminus S$ is π_1 -injective and $\pi_1(B)$ is a lattice in the universal cover of $Spin(n-1, 2)$. The proof in [Tol93] is written under the simplifying assumptions that $n \geq 4$ and n is even, in which case Raghunathan's work [Rag84] implies that $\pi_1(B)$ is not residually finite. Since residual finiteness of a group is inherited by subgroups, it followed that $\pi_1(M \setminus S)$ is not residually finite. Toledo comments that the above proof also works when (M, S) is modelled on $(\mathbf{CH}^n, \mathbf{CH}^{n-1})$ except that [Rag84] is not available. Raghunathan's work is intimately related with solution of the congruence subgroup problem for $SO(n-1, 2)$, which is wide open for lattices in $SU(n-1, 1)$.

If (M, S) is modelled on $(\mathbf{CH}^n, \mathbf{CH}^{n-1})$, then there is a strong motivation for trying to show that $\pi_1(M \setminus S)$ need not be residually finite. Indeed, by Part (5) of Theorem 1.4, which is based on Osin's Dehn Surgery theorem [Osi07], $\pi_1(M \setminus S)$ is *residually hyperbolic*, and therefore, if $\pi_1(M \setminus S)$ is *not* residually finite for some (M, S) , then there exists a hyperbolic group that is not residually finite. Of course, there are many residually hyperbolic groups that do not look residually finite, and the main reason I single out $\pi_1(M \setminus S)$ as a candidate for disproving residual finiteness of hyperbolic groups is that $\pi_1(M \setminus S)$ is not far from being a lattice so perhaps its finite index subgroups could be sometimes understood via arithmetic means.

Another promising candidate is $\pi_1(B)$, where as before B is the boundary of a tubular neighborhood of a component of S . Indeed, $\pi_1(B)$ is a lattice in the universal cover of $SU(n-1, 1)$, which is a nonlinear semisimple Lie group (see e.g. [ABC⁺96, page 115]). As noted in Lemma 13.1, B is a circle bundle over a component of S whose first Chern class is the $-\frac{1}{4\pi}$ -multiple of the Kähler form of S , and therefore, $\pi_1(B)$ is an extension with infinite cyclic kernel and hyperbolic quotient, which does not virtually split. By a straightforward argument, any extension with infinite cyclic kernel and hyperbolic quotient is residually hyperbolic, yet it is unclear whether $\pi_1(B)$ is always residually finite.

In the same circle of ideas belongs a remark of Lubotzky [Lub05] that if one could prove the congruence subgroup property for a lattice in $Sp(n, 1)$ or $F_4^{(-20)}$, then this lattice has a hyperbolic quotient which is not residually finite.

2. OUTLINE OF THE CURVATURE COMPUTATION

In the context of this paper a multiply-warped product is a metric of the form $dr^2 + g_r$ where r varies in an open interval and g_r is a family of Riemannian metric on a smooth manifold F constructed by fixing a Riemannian metric \mathbf{f} on F , considering an orthogonal splitting of the tangent bundle TF into (possibly nonintegrable) subbundles H_i , and scaling the metric on each H_i by a warping function $h_i = h_i(r)$. The key issues in constructing multiply-warped metrics with prescribed curvature bounds are

- (1) to come up with curvature formulas such that the bounds on curvature translate into *simple* differential inequalities on warping functions h_i ,
- (2) to construct warping functions h_i that satisfy the inequalities.

Part (1) depends on the specifics of the geometry of (F, \mathbf{f}) and on interaction between H_i 's, e.g. the curvature formulas for (F, g_r) typically involve brackets of vector fields from different H_i 's, and if each H_i is integrable, the formulas simplify considerably. Part (2) is driven by the shape of the differential inequalities obtained in Part (1). The methods used in Part (2) are usually those of single variable calculus and elementary ODE, yet making them work is a specialized craft involving a number of tricks, and the intuition behind the tricks is intimately related to the geometry of the desired curvature bound, be that negative, almost nonnegative, or Ricci positive curvature.

In Section 3 we write the complex hyperbolic metric on the ends of $M \setminus S$ in cylindrical coordinates about S as

$$dr^2 + \sinh^2(r)d\theta^2 + \cosh^2\left(\frac{r}{2}\right)\mathbf{k}^{n-1}$$

where r is the distance to S , and θ is the parameter on the unit circle about S , and \mathbf{k}^{n-1} is the complex hyperbolic metric S . The “+” refers to the orthogonal splitting of the tangent bundle to $M \setminus S$ into the sum of integrable subbundles spanned by ∂_r and $\frac{\partial}{\partial \theta}$, and their orthogonal complement \mathcal{H} , which is nonintegrable. We then modify the metric on the ends of $M \setminus S$ to be

$$\lambda_{v,h} := dr^2 + v^2 d\theta^2 + h^2 \mathbf{k}^{n-1}$$

and compute its curvature tensor in terms of v, h , where v, h are positive functions of r , which varies from $-\infty$ to the normal injectivity radius of S . Formulas of Appendix B reduce the problem to computing curvatures of the r -tubes about S . In Section 4 we set up a convenient frame in which the curvature tensor components are to be computed. The “structure constants” of brackets in the frame are computed in Section 5 by specializing to the complex hyperbolic space where all curvatures are known. Each r -tube about S comes with the Riemannian submersion metric $v^2 d\theta^2 + h^2 \mathbf{k}^{n-1}$ which has totally geodesic circle fibers, so we use O’Neill’s formulas to compute the curvature tensor of the tube.

This is done in Sections 7–8 where we also arrange for several computational simplifications, notably, we shall never need to know $\langle R(X_i, X_j), X_k, X_1 \rangle$ where X_1 is vertical and X_i, X_j, X_k are linearly independent horizontal vector fields. Putting all this together in Section 9, we obtain a reasonably simple formulas for the sectional curvature.

In Section 10 we choose v, h so that $M \setminus S$ becomes complete, finite volume, and of sectional curvature bounded above by a negative constant, and furthermore the metric is complex hyperbolic away from a small tubular neighborhood of S . This is the heart of the proof, and to help digesting it we outline what we shall do, and why we do it.

First of all, we assume that v, h are positive so that the metric is nondegenerate. After glancing over the curvature formulas (9.2)–(9.5) it is apparent that we need h'', v'' to be positive, and furthermore, h', v' may not vanish for if $h'v' = 0$, then $K(Y_2, Y_1) > 0$. This means that v, h are increasing everywhere as they are equal to increasing functions $\sinh(r)$, $\cosh(r/2)$ for sufficiently large r . As a starting point, we let $h(r) = e^{r/2}$ and $v(r) = \epsilon e^r$ on a neighborhood of $-\infty$, where $0 < \epsilon \ll 1$ is a parameter, and for these h, v it is easy to compute that $\sec(\lambda_{v,h}) < -\frac{1}{10}$. The main issue is to interpolate v, h in between while keeping curvature negative.

The graphs of $\sinh(r)$ and ϵe^r intersect at a point $r_\epsilon \approx \epsilon$, and v is obtained from the (strictly convex) function $\max\{\sinh(r), \epsilon e^r\}$ by smoothing it near r_ϵ so that $v(r) = \epsilon e^r$ for $r \leq r_\epsilon - 2\epsilon^4$ and $v(r) = \sinh(r)$ for $r \geq r_\epsilon + 2\epsilon^4$. While smoothing we need to be able to estimate $\frac{v'}{v}$, and also need to keep a definite lower bound on $\frac{v''}{v}$. This is accomplished by making v satisfy $(\ln(v))'' > 0$ over the smoothing region, so that $\frac{v''}{v} > \left(\frac{v'}{v}\right)^2$, and $\frac{v'}{v}$ is increasing, which allows us to estimate $\frac{v'}{v}$ by its values at the endpoints.

Then we construct h by bending down the graph of $\cosh(r/2)$ near $r = \epsilon/2$ so that it eventually agrees with $e^{r/2}$. The tangent line to the graph of $\cosh(r/2)$ near $\epsilon/2$ is almost horizontal, and it intersects the graph of $e^{r/2}$ near $r = -\frac{8}{\epsilon}$, and thus we bend $\cosh(\frac{r}{2})$ over the interval $[-\frac{8}{\epsilon}, \frac{\epsilon}{2}]$; note that on this interval $v = \epsilon e^r$. Bending h is done in two stages, which helps to control $\frac{h'}{h}$; during the first stage h almost coincides with the tangent line to $\cosh(r/2)$ at $\epsilon/2$, and during the second stage we bend h upwards, so that $(\ln(h))'' > 0$. At either stage we manage to estimate $\frac{h''}{h}$, $\frac{h'}{h}$.

In fact, for technical reasons we build v, h by first producing “easy-to-visualize” C^1 functions \mathbf{v}, \mathbf{h} , which we then smooth via convolutions to get good lower bounds on the second derivatives of v, h using Appendix A.

Finally, we estimate the curvature of $\lambda_{v,h}$ over two disjoint (!) regions, one where v is bent, and the other where h is bent. The main difficulty is to

control the “mixed” term (9.5), and it turns out that the terms $K(Y_3, Y_2)$, $K(\partial_r, Y_1)$, $K(\partial_r, Y_2)$ in formulas (9.2)–(9.4) carry enough negative curvature to compensate the positivity of the “mixed” term. Over the region where h is bent, $\frac{h'}{h}$, $\frac{v'}{v}$ are kept bounded and $v = \epsilon e^r$, so if ϵ is small, then $\frac{v}{h^2}$ becomes small when $\epsilon \rightarrow 0$, and hence the “mixed” term is negligible. On the other hand, over the region where v is bent, the “mixed” term does not become small, and instead it is compensated by $K(\partial_r, Y_1)$ and $K(Y_3, Y_2)$, and the estimate hinges on how c_{23} enters in $K(Y_3, Y_2)$ and in the “mixed” term.

The proof takes several pages of tedious curvature estimates, which seems hard to shorten. Linear algebra arguments throughout this proof are repetitive, and it is conceivable that they could be simplified by doing the computation in a different frame, e.g. the one that diagonalizes the curvature operator. Unfortunately, it seems that this would make the formulas for the sectional curvature of the coordinate planes much more complicated than those in formulas (9.2)–(9.5), so at the end we would gain nothing.

After proving that sectional curvature is bounded above by a negative constant, we apply a result in [Bel] to check that $\pi_1(M \setminus S)$ -action on the universal cover of $M \setminus S$ satisfies Gromov’s definition of relative hyperbolicity, which proves Part (i) of Theorem 1.1.

Part (ii) is proved in Section 11. We keep $v = \epsilon e^r$, and bend the function h constructed above near $-\infty$ so that there it becomes equal to $\tau_\epsilon + e^{r/2}$, where τ_ϵ is a carefully chosen positive constant. By formulas (9.2)–(9.5), this choice of h ensures that the sectional curvature is bounded, and following the pattern of Part (i) we prove that the curvature is negative (but not bounded away from zero, which would be impossible by Part (13) of Theorem 1.4). Furthermore, using formulas (9.2)–(9.5), and the fact that $v = \epsilon e^r$, $h = \tau_\epsilon + e^{r/2}$ near $-\infty$, we are able to show that all the derivatives of the curvature tensor have bounded components, so the metric is A -regular.

3. COMPLEX HYPERBOLIC SPACE IN CYLINDRICAL COORDINATES

We follow [Gol99] for conventions and background on complex hyperbolic geometry. In particular, the complex hyperbolic space \mathbf{CH}^n is normalized to have holomorphic sectional curvature -1 , and we denote the complex hyperbolic metric by \mathbf{k}^n , or simply by \mathbf{k} for brevity.

The purpose of this section is to describe “cylindrical coordinates” on \mathbf{CH}^n about a complex hyperplane \mathbf{CH}^{n-1} . The boundary of the r -neighborhood of \mathbf{CH}^{n-1} is a real hypersurface, which is denoted by $F(r)$, and is referred to as an r -tube. Thus the metric on \mathbf{CH}^n can then be written as $\mathbf{k} = dr^2 + \mathbf{k}_r$ where \mathbf{k}_r is the induced Riemannian metric on $F(r)$, and we need to describe \mathbf{k}_r . This computation seems to be unknown to experts, so we give full details.

The orthogonal projection $\pi: \mathbf{CH}^n \rightarrow \mathbf{CH}^{n-1}$ is a fiber bundle whose fibers are complex geodesics, i.e. totally geodesic complex submanifolds isometric to the real hyperbolic plane of curvature -1 [Gol99, Theorem 3.1.9]. Restricting π to $F(r)$ gives a circle bundle $\pi_r: F(r) \rightarrow \mathbf{CH}^{n-1}$ whose fiber over $w \in \mathbf{CH}^{n-1}$ is the circle of radius r in the complex geodesic $\pi^{-1}(w)$. The tangent bundle $TF(r)$ splits orthogonally as $\mathcal{V}(r) \oplus \mathcal{H}(r)$, where $\mathcal{V}(r)$ is tangent to the circle $\pi^{-1}(w) \cap F(r)$, and $\mathcal{H}(r)$ is the orthogonal complement of $\mathcal{V}(r)$. Thus any vector in $TF(r)$ can be uniquely decomposed as $V + H$, where $V \in \mathcal{V}(r)$ and $H \in \mathcal{H}(r)$, and

$$\mathbf{k}_r(V + H, V + H) = \mathbf{k}_r(V, V) + \mathbf{k}_r(H, H).$$

As we explain below under suitable identifications, the restriction of \mathbf{k}_r to $\mathcal{V}(r)$ is $\sinh^2(r)d\theta^2$, and the restriction of \mathbf{k}_r to $\mathcal{H}(r)$ is $\cosh^2(\frac{r}{2})\mathbf{k}^{n-1}$, where $d\theta^2$ is the standard metric on the unit circle \mathbf{S}^1 .

As \mathbf{CH}^n has nonpositive sectional curvature, and \mathbf{CH}^{n-1} is totally geodesic, the hyperplane \mathbf{CH}^{n-1} has infinite normal injectivity radius, and the map $r: \mathbf{CH}^n \setminus \mathbf{CH}^{n-1} \rightarrow (0, \infty)$ is a (smooth) Riemannian submersion with fibers $F(r)$. The geodesic flow along *radial* (i.e. orthogonal to \mathbf{CH}^{n-1}) geodesics induces a diffeomorphism between different tubes $\phi_{sr}: F(s) \rightarrow F(r)$, $s, r > 0$, and also preserves every complex geodesics orthogonal to \mathbf{CH}^{n-1} . As we prove below, the differential $d\phi_{sr}$ maps $H(s)$ to $H(r)$, and $V(s)$ to $V(r)$. Fix an arbitrary radial unit speed geodesic $\gamma(r)$ with $\gamma(0) = w \in \mathbf{CH}^{n-1}$.

That $d\phi_{sr}$ takes $\mathcal{V}(s)$ to $\mathcal{V}(r)$ is obvious because $\mathcal{V}(r)$ is tangent both to $F(r)$ and the complex geodesic $\pi^{-1}(w)$, thus ϕ_{sr} restricted to $\pi^{-1}(w)$ simply maps the s -circle centered at w to the concentric r -circle, whose tangent bundles are $\mathcal{V}(s)$, $\mathcal{V}(r)$, respectively. Since $\pi^{-1}(w)$ is a hyperbolic plane of curvature -1 , its metric can be written as $dr^2 + \sinh^2(r)d\theta^2$ where $d\theta^2$ is the standard metric on the unit circle, so that the metric on $\mathcal{V}(r)$ equals to $\sinh^2(r)d\theta^2$. In the (r, θ) -coordinates the map ϕ_{sr} becomes $(s, \theta) \rightarrow (r, \theta)$ because the lines $\theta = \text{constant}$ are geodesics, and therefore, the vector field $\frac{\partial}{\partial \theta}$ is $d\phi_{sr}$ -invariant.

Let δ be a unit speed geodesic in \mathbf{CH}^{n-1} with $\sigma(0) = w$. By [Gol99, Lemma 3.2.13] the exponential map takes the plane $\text{span}(\gamma', \delta') \subset T_w \mathbf{CH}^{n-1}$ to a totally real (and hence totally geodesic) 2-plane R_δ in \mathbf{CH}^n which intersects \mathbf{CH}^{n-1} along δ and intersects the complex geodesic $\pi^{-1}(w)$ along γ . If t denotes the arclength parameter on δ , then the metric on R_δ can be written in the (r, t) -coordinates as $dr^2 + q^2(r, t)dt^2$ where $r, t \in \mathbb{R}$, and in fact $q(r, t)$ is independent of t because the isometric \mathbb{R} -action on R_δ by translations along δ extends to an isometric \mathbb{R} -action on \mathbf{CH}^n . Since the sectional curvature of any totally real plane is $-1/4$ [Gol99, page 80], and since the sectional curvature of $dr^2 + q^2(r)dt^2$ is $-\frac{q''}{q}$, we conclude that $q(r) = \cosh(\frac{r}{2})$.

Being totally geodesic, R_δ is preserved by the geodesic flow, and in the (r, t) -coordinates ϕ_{sr} maps (s, t) to (r, t) for $s, r \geq 0$. In particular, the vector field $\frac{\partial}{\partial t}$ is $d\phi_{sr}$ -invariant, i.e. $d\phi_{sr}(\frac{\partial}{\partial t}) = \frac{\partial}{\partial t}$. It follows that $d\phi_{sr}(\frac{\partial}{\partial t}) \in \mathcal{H}$; indeed $\frac{\partial}{\partial t}$ is clearly orthogonal to ∂_r , and $\frac{\partial}{\partial t}$ is orthogonal to $\frac{\partial}{\partial \theta} = J\partial_r$ because the span of $\frac{\partial}{\partial t}$, ∂_r is totally real.

Since every vector in $T_w \mathbf{CH}^{n-1}$ is proportional to some $\sigma'(0) = \frac{\partial}{\partial t}$, and $\frac{\partial}{\partial t}$ is $d\phi_{sr}$ -invariant, the linear maps $d\phi_{0r}: T_w \mathbf{CH}^{n-1} \rightarrow \mathcal{H}(r)$ and $d\phi_{sr}: \mathcal{H}(s) \rightarrow \mathcal{H}(r)$, are injective, and hence they must be isomorphisms by dimension reasons.

The length of $\frac{\partial}{\partial t} \in TR_\delta \subset \mathcal{H}(r)$ is $\cosh(\frac{r}{2})$, so the metric on $\mathcal{H}(r)$ can be written as $\cosh^2(\frac{r}{2})\mathbf{k}^{n-1}$, or more explicitly, for $H \in \mathcal{H}(r)$

$$\mathbf{k}_r(H, H) = \cosh^2\left(\frac{r}{2}\right) \mathbf{k}^{n-1}(d\phi_{0r}^{-1}(H), d\phi_{0r}^{-1}(H)).$$

Thus the metric q_r on $TF(r) = \mathcal{V}(r) \oplus \mathcal{H}(r)$ can be written as $\sinh^2(r)d\theta^2 + \cosh^2(\frac{r}{2})\mathbf{k}^{n-1}$. It is clear that $\sinh^2(r)d\theta^2 + \cosh^2(\frac{r}{2})\mathbf{k}^{n-1}$ is a Riemannian submersion metric whose base is the $\cosh(\frac{r}{2})$ multiple of \mathbf{CH}^{n-1} , and fibers are standard circles of radius $\sinh(r)$.

Finally, fix an arbitrary tube, denote it by F , and use the diffeomorphisms ϕ_{sr} to pull back $\mathcal{V}(r)$, $\mathcal{H}(r)$, \mathbf{k}_r to F . Since the pullbacks of $\mathcal{V}(r)$, $\mathcal{H}(r)$ are independent of r , we just denote the corresponding subbundles of TF by \mathcal{H} , \mathcal{V} . As $\pi \circ \phi_{sr} = \pi$, the projections $\pi_r: F(r) \rightarrow \mathbf{CH}^{n-1}$ all get identified via ϕ_{sr} to a circle bundle projection $F \rightarrow \mathbf{CH}^{n-1}$ whose differential takes \mathcal{V} to zero, and maps \mathcal{H} onto $T\mathbf{CH}^{n-1}$. In summary, the complex hyperbolic manifold $\mathbf{CH}^n \setminus \mathbf{CH}^{n-1}$ is now written as $(0, \infty) \times F$ equipped with the metric

$$dr^2 + \sinh^2(r)d\theta^2 + \cosh^2\left(\frac{r}{2}\right) \mathbf{k}^{n-1}.$$

4. BASIS AND BRACKETS

We borrow the notations $F, \mathcal{H}, \mathcal{V}, k_n$ from Section 3, and fix an open interval I . Given positive smooth functions v, h on I , let $\lambda_{v,h,r}$ be the Riemannian submersion metric on $TF = \mathcal{V} \oplus \mathcal{H}$ with base $h\mathbf{CH}^{n-1}$ and fiber $v\mathbf{S}^1$; we also write

$$\lambda_{v,h,r} := v^2 d\theta^2 + h^2 \mathbf{k}^{n-1}.$$

This gives rise to the metric $\lambda_{v,h} = dr^2 + \lambda_{v,h,r}$ on $I \times F$. For brevity we sometimes suppress v, h and label tensors associated with $\lambda_{v,h}$, $\lambda_{v,h,r}$ by λ , λ_r , respectively.

Example 4.1. If $I = (0, \infty)$, $v = \sinh(r)$ and $h = \cosh(\frac{r}{2})$, then $\lambda_{v,h,r} = \mathbf{k}_r$ so that $\lambda_{v,h} = dr^2 + \mathbf{k}_r = \mathbf{k}^n$ is the complex hyperbolic metric.

The purpose of this section is to introduce a convenient local orthonormal frame on $I \times F$ in which the curvature of $\lambda_{v,h}$ will be computed. To this end denote $\frac{\partial}{\partial r}$ by ∂_r , and $\frac{\partial}{\partial \theta}$ by X_1 . Fix $z \in I \times F$ and let $w \in \mathbf{CH}^{n-1}$ be the image of z under the map $p: I \times F \rightarrow \mathbf{CH}^{n-1}$ obtained by composing the projection to the second factor $I \times F \rightarrow F$ with the circle bundle $F \rightarrow \mathbf{CH}^{n-1}$.

Let $\{\check{X}_i\}$, with $1 < i < 2n$, be an arbitrary orthonormal frame defined on a neighborhood of w in \mathbf{CH}^{n-1} such that $[\check{X}_i, \check{X}_j]$ vanishes at w . (By a standard argument any orthonormal basis in $T_w \mathbf{CH}^{n-1}$ can be extended to some $\{\check{X}_i\}$ as above). Let X_i be the vector field obtained by lifting \check{X}_i to a horizontal vector field in $\mathcal{H} \subset TF$, and then pullbacking it via the projection $I \times F \rightarrow F$. Then $\partial_r, X_1, \dots, X_{2n-1}$ is an orthogonal frame near z such that

- (1) $\langle X_1, X_1 \rangle_\lambda = v^2$, and $\langle X_i, X_i \rangle_\lambda = h^2$ for $i > 1$.
- (2) $[X_i, X_j]$ is tangent to level surfaces of r ,
- (3) $[X_i, \partial_r] = 0$ because X_i is invariant under the flow of ∂_r ,
- (4) $[X_i, X_1] = 0$ because X_i is invariant under the flow of X_1 on F that corresponds to the rotation about \mathbf{CH}^{n-1} in \mathbf{CH}^n .
- (5) $[X_i, X_j]$ is vertical at z because $[\check{X}_i, \check{X}_j]$ vanishes at w ; thus there exists “structure constants” $c_{ij} \in \mathbb{R}$ with $[X_i, X_j] = c_{ij} X_1$ at z .

The corresponding orthonormal frame $\partial_r, Y_1 = \frac{1}{v} X_1, Y_i = \frac{1}{h} X_i, i > 1$ enjoys the following properties:

- (i) $[Y_i, Y_j] = \frac{1}{h^2} [X_i, X_j] = c_{ij} \frac{v}{h^2} Y_1$ for $i, j > 1$,
- (ii) $[Y_i, Y_1] = \frac{1}{hv} [X_i, X_1] = 0$,
- (iii) $[Y_1, \partial_r] = \frac{v}{h} Y_i$, and $[Y_i, \partial_r] = \frac{h'}{h} Y_i$ for $i > 1$,

where the first equalities in (i), (ii) hold because any function of r has zero derivative in the direction of X_i , and (iii) follows from $[X_i, \partial_r] = 0$.

5. A CURVATURE FORMULA IN THE COMPLEX HYPERBOLIC SPACE

In Appendix B we explain, following [BW04, Bel], how to relate the components of the curvature tensors of $\lambda_{v,h}$ and $\lambda_{v,h,r}$ in the basis $\{\partial_r, Y_1, \dots, Y_{2n-1}\}$, but more work is needed to compute these components in terms of v, h only, and this section provides one of the main steps in the computation. Specifically, we compute the components of the curvature tensor of \mathbf{k}^n in the basis $\{\partial_r, Y_1, \dots, Y_{2n-1}\}$, and establish some useful identities on the structure constants c_{ij} .

Convention: In this section $\langle \cdot, \cdot \rangle$, R , J denote the metric, the curvature tensor, and the complex structure on \mathbf{CH}^n , respectively; in other words, we suppress the subscript \mathbf{k} for all tensors throughout the section.

In the complex hyperbolic space one has the following explicit formula for the $(4,0)$ -curvature tensor R in terms of \mathbf{k} and J (see [KN96, Proposition IX.7.3]): for any tangent vectors X, Y, Z, W

$$4\langle R_k(X, Y)Z, W \rangle = \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle + \langle X, JZ \rangle \langle Y, JW \rangle - \langle X, JW \rangle \langle Y, JZ \rangle + 2\langle X, JY \rangle \langle Z, JW \rangle.$$

If X, Y, Z, W belongs to the \mathbf{k} -orthonormal basis $\{\partial_r, Y_1, \dots, Y_{2n-1}\}$ with $X_1 = Y_1 \sinh(r)$ and $X_i = Y_i \cosh(\frac{r}{2})$ for $i > 1$, we get the following:

$$(5.1) \quad \langle R_k(\partial_r, Y_1)Y_i, Y_j \rangle = \frac{1}{2}\langle \partial_r, JY_1 \rangle \langle Y_i, JY_j \rangle = -\frac{1}{2}\langle Y_i, JY_j \rangle \quad \text{if } i, j > 1,$$

$$(5.2) \quad \langle R_k(Y_i, Y_j)Y_j, Y_k \rangle = 0 \quad \text{if } i, j, k \text{ are distinct,}$$

$$(5.3) \quad \langle R_k(Y_i, Y_j)Y_j, Y_i \rangle = -\frac{1}{4} - \frac{3}{4}\langle Y_i, JY_j \rangle^2 \quad \text{if } i \neq j.$$

In computing (5.1) the first two summands of $4\langle R_k(X, Y)Z, W \rangle$ vanish because $\{\partial_r, Y_1, \dots, Y_{2n-1}\}$ is orthonormal, and the next two summands vanish because $\langle \partial_r, JY_k \rangle = 0$ unless $k = 1$. The last equality in (5.1) is true because $\langle \partial_r, JY_1 \rangle = -1$ with respect to the standard complex structure.

In computing (5.2), the first two summands vanish as i, j, k are distinct and $\{Y_i\}$ is orthonormal. The forth summand vanishes because the metric is Hermitian so that Y_j, JY_j are orthogonal. Finally, $\langle Y_i, JY_j \rangle \langle Y_j, JY_k \rangle = 0$ because \mathcal{V} (and hence \mathcal{H}) are J -invariant and orthogonal, so the only way $\langle Y_i, JY_j \rangle$ could be nonzero is when $k = 1$, but then $\langle Y_j, JY_k \rangle = 0$.

In computing (5.3) the first summand vanishes and the second summand is -1 because $\{Y_i\}$ is orthonormal. The forth summand vanishes as $\langle Y_i, JY_i \rangle = 0$, and the other two summands may survive and add up to the promised expression since $\langle Y_i, JY_j \rangle = -\langle JY_i, Y_j \rangle$.

Lemma 5.4. $\langle Y_i, JY_j \rangle = 2c_{ij}$.

Proof. The result follows by combining (5.1) with

$$(5.5) \quad \langle R_k(\partial_r, Y_1)Y_i, Y_j \rangle = \langle [Y_j, Y_i], Y_1 \rangle \left(\ln \frac{v}{h} \right)' = -c_{ij} \frac{v}{h^2} \left(\ln \frac{v}{h} \right)' = -c_{ij},$$

where the first equality comes from Appendix B and $[Y_i, Y_1] = 0$, the second equality follows from $[Y_i, Y_j] = c_{ij} \frac{v}{h^2} Y_1$, and the last equality holds by explicit computation with $v = \sinh(r)$ and $h = \cosh(\frac{r}{2})$. \square

Remark 5.6. The metric \mathbf{k} is Hermitian, hence $\langle Y_i, JY_j \rangle^2 \leq 1$. Therefore,

$$(5.7) \quad |\langle R_k(\partial_r, Y_1)Y_i, Y_j \rangle| = |c_{ij}| \leq \frac{1}{2},$$

$$(5.8) \quad \langle R_k(Y_i, Y_j)Y_j, Y_i \rangle = -\frac{1}{4} - 3c_{ij}^2 \in [-\frac{1}{4}, -1] \quad \text{for } i \neq j.$$

Lemma 5.9. If $i \neq j$, then $c_{ij} = 0$ iff Y_i, Y_j span a totally real plane.

Proof. Recall that a plane is called *totally real* if it is orthogonal to its J -image. Since \mathbf{k} is Hermitian, Y_i is always orthogonal to JY_i . Hence, the span of Y_i, Y_j is orthogonal to its image iff Y_i and JY_j are orthogonal, which by Lemma 5.4 is equivalent to $c_{ij} = 0$. \square

Example 5.10. By (ii) of Section 4, $c_{i1} = 0$. (Alternatively $\langle Y_i, JY_1 \rangle = 0$ for $i > 1$ because \mathcal{V}, \mathcal{H} are J -invariant, and $\langle Y_1, JY_1 \rangle = 0$ as \mathbf{k} is Hermitian). Thus (5.3) implies that $\langle R_k(Y_i, Y_1)Y_1, Y_i \rangle = -\frac{1}{4}$ for $i > 1$, which by [Gol99, Lemma 3.2.13] also follows from the fact that $\{Y_1, Y_i\}$ spans a totally real plane.

Example 5.11. By Lemma 5.4, $|c_{ij}| = \frac{1}{2}$ iff $Y_i = \pm JY_j$, which in turn is equivalent to saying that $\{Y_i, Y_j\}$ spans a complex geodesic, i.e. a totally geodesic complex line. Note that complex geodesics have sectional curvature -1 . If $n = 2$, then \mathcal{H} has complex dimension one, which forces $Y_3 = \pm JY_2$ so that $c_{23} = \pm \frac{1}{2}$.

Lemma 5.12. $\langle R_k(\check{X}_i, \check{X}_j)\check{X}_j, \check{X}_i \rangle = -\frac{1}{4} - 3c_{ij}^2$.

Proof. The vector fields Y_i , $i > 1$ were defined for $r > 0$, yet they smoothly extend to points with $r = 0$, and in fact if $r = 0$, then $Y_i = X_i = \check{X}_i$ as $\cosh(0) = 1$. By continuity the formula (5.8) still holds for $r = 0$ giving the promised result. \square

Remark 5.13. Since \mathcal{H} is J -invariant, $JY_i \in \mathcal{H}$ for $i > 1$, so writing Y_i in the orthonormal basis $\{Y_j\}$, $j > 1$ of \mathcal{H} , we get

$$JY_i = \sum_{j>1} \langle JY_i, Y_j \rangle Y_j = -2 \sum_{j>1} c_{ij} Y_j,$$

and since $|JY_i|_k = 1$, we obtain the identity

$$(5.14) \quad \sum_{j>1} c_{ij}^2 = \frac{1}{4}.$$

6. COMPUTING A AND T TENSORS

In this section we compute the A and T tensors of the metric

$$q_{v,h,r} := \frac{1}{h^2} \lambda_{v,h,r} = \frac{v^2}{h^2} d\theta^2 + \mathbf{k}^{n-1},$$

which is a Riemannian submersion metric with base \mathbf{CH}^{n-1} and fiber $\frac{v}{h} \mathbf{S}^1$. We often suppress v, h and denote $q_{v,h,r}$ by q_r , and the associated norm by $|\cdot|_{q_r}$.

First show that the fibers of the Riemannian submersion are totally geodesic i.e. $T = 0$. (Indeed, rescaling takes geodesics to geodesics, it suffices to give

a proof for $\lambda_{v,h,r}$. Then we need to show that $\nabla_{Y_1} Y_1$ is proportional to ∂_r , which is obvious because $\nabla_{Y_1} Y_1$ is tangent to the complex geodesic orthogonal to \mathbf{CH}^{n-1} , and $\nabla_{Y_1} Y_1$ is orthogonal to Y_1 because Y_1 has λ_r -length one).

Next compute the A tensor of q_r at the point z in the basis $\{X_i\}$. If $i, j > 1$, then $A_{X_i} X_j$ is the q_r -orthogonal projection of $\frac{1}{2}[X_i, X_j] = \frac{1}{2}c_{ij}X_1$ to \mathcal{V} , hence $A_{X_i} X_j = \frac{c_{ij}}{2}X_1$. Since $|X_1|_{q_r} = \frac{v}{h}$, we get for $i, j > 1$ that $|A_{X_i} X_j|_{q_r} = \frac{|c_{ij}|}{2} \frac{v}{h}$, and in particular, $|A_{X_i} X_j|_{q_r} \leq \frac{v}{4h}$. By [Bes87, 9.21d]

$$\langle A_{X_i} X_1, X_j \rangle_{q_r} = -\langle A_{X_i} X_j, X_1 \rangle_{q_r} = -\frac{v^2}{h^2} \frac{c_{ij}}{2}.$$

Since $\{X_i\}$ is a q_r -orthonormal basis in \mathcal{H} and $A_{X_i} X_1$ is horizontal, we get

$$A_{X_i} X_1 = -\frac{v^2}{2h^2} \sum_{j>1} c_{ij} X_j, \text{ so that } |A_{X_i} X_1|_{q_r} = \frac{v^2}{2h^2} \sqrt{\sum_{j>1} c_{ij}^2} = \frac{v^2}{4h^2}$$

where the second equality in the latter formula follows from (5.14). With the A tensor is computed, O'Neill's formulas [Bes87, Theorem 9.28] allow us to calculate the sectional curvature.

$$(6.1) \quad \langle R_{q_r}(X_1, X_i)X_i, X_1 \rangle_{q_r} = |A_{X_i} X_1|_{q_r}^2 = \frac{v^4}{16h^4}$$

$$(6.2) \quad \langle R_{q_r}(X_i, X_j)X_j, X_i \rangle_{q_r} = -\frac{1}{4} - 3c_{ij}^2 - 3c_{ij}^2 \frac{v^2}{4h^2}$$

where (6.2) depends on Lemma 5.12.

7. SECTIONAL CURVATURES OF COORDINATE PLANES

Since the $(4, 0)$ -curvature tensor scales like the metric, we get for $i > 1$

$$\langle R_{\lambda_r}(Y_1, Y_i)Y_i, Y_1 \rangle_{\lambda_r} = \frac{h^2}{v^2 h^2} \langle R_{q_r}(X_1, X_i)X_i, X_1 \rangle_{q_r}$$

and hence formulas (6.1), (B.1) imply

$$(7.1) \quad \langle R_{\lambda}(Y_1, Y_i)Y_i, Y_1 \rangle_{\lambda} = \frac{v^2}{16h^4} - \frac{v'}{v} \frac{h'}{h}.$$

Similarly, for distinct $i, j > 1$

$$\langle R_{\lambda_r}(Y_i, Y_j)Y_j, Y_i \rangle_{\lambda_r} = \frac{h^2}{h^4} \langle R_{q_r}(X_i, X_j)X_j, X_i \rangle_{q_r}$$

so formulas (6.2), (B.1) imply

$$(7.2) \quad \langle R_{\lambda}(Y_i, Y_j)Y_j, Y_i \rangle_{\lambda} = -\frac{1}{4h^2} - \frac{3}{h^2} c_{ij}^2 - 3c_{ij}^2 \frac{v^2}{4h^4} - \left(\frac{h'}{h}\right)^2.$$

and also by (B.1)

$$(7.3) \quad \langle R_{\lambda}(Y_i, \partial_r)\partial_r, Y_i \rangle_{\lambda} = -\frac{h''}{h}, \quad \langle R_{\lambda}(Y_1, \partial_r)\partial_r, Y_1 \rangle_{\lambda} = -\frac{v''}{v}.$$

Remark 7.4. A wary reader may wish to play with trigonometric identities to verify the formulas (7.1), (7.2), (7.3) in the complex hyperbolic case, where for $i > j \geq 1$ the planes spanned by $\{Y_i, \partial_r\}$, or by $\{Y_i, Y_j\}$ with $c_{ij} = 0$ are always totally real, and hence their sectional curvature is $-\frac{1}{4}$, while $\{Y_1, \partial_r\}$, or $\{Y_i, Y_j\}$ with $c_{ij} = \pm\frac{1}{2}$ span complex geodesics of curvature -1 .

8. MIXED COMPONENTS OF THE CURVATURE TENSOR

As we show in Section 9 to compute the sectional curvatures of λ one only needs to know the components of R_λ involving $\partial_r, Y_1, Y_2, Y_3$. In this section we show that all the *mixed* components involving $\partial_r, Y_1, Y_2, Y_3$ vanish except for those listed (up to symmetries of the curvature tensor) in (8.1), (8.2) below.

First, note that by Appendix B we have $\langle R_g(Y_i, \partial_r)\partial_r, Y_j \rangle = 0$ if $i \neq j$, and since $[Y_i, Y_j] = c_{ij}\frac{v}{h^2}Y_1$ and $c_{i1} = 0 = c_{1j}$, the only terms of the form $\langle R_\lambda(\partial_r, Y_i)Y_j, Y_k \rangle_\lambda$ that could be nonzero at z are as follows (up to symmetries of the curvature tensor):

$$(8.1) \quad \langle R_\lambda(\partial_r, Y_1)Y_i, Y_j \rangle_\lambda = \langle [Y_j, Y_i], Y_1 \rangle_\lambda \left(\ln \frac{v}{h}\right)' = -c_{ij}\frac{v}{h^2} \left(\ln \frac{v}{h}\right)',$$

$$(8.2) \quad 2\langle R_\lambda(\partial_r, Y_i)Y_j, Y_1 \rangle_\lambda = \langle [Y_i, Y_j], Y_1 \rangle_\lambda \left(\ln \frac{v}{h}\right)' = c_{ij}\frac{v}{h^2} \left(\ln \frac{v}{h}\right)',$$

where $i, j > 1$ and $i \neq j$. The remaining mixed components involving $\partial_r, Y_1, Y_2, Y_3$ vanish by the following.

Lemma 8.3. *If i, j, k are distinct, then $\langle R_\lambda(Y_i, Y_j)Y_j, Y_k \rangle = 0$.*

Proof. In this paper we only use this lemma when $\{i, j, k\} = \{1, 2, 3\}$, but proving the general case is not much harder. The idea of the proof is to show that $\langle R_\lambda(Y_i, Y_j)Y_j, Y_k \rangle_\lambda$ is proportional to $\langle R_k(Y_i, Y_j)Y_j, Y_k \rangle_k$ which is zero by (5.2). By the formula (B.1) and the fact that the curvature tensor scales like the metric we have

$$\langle R_\lambda(Y_i, Y_j)Y_j, Y_k \rangle_\lambda = \langle R_{\lambda_r}(Y_i, Y_j)Y_j, Y_k \rangle_{\lambda_r} = \frac{1}{h^2} \langle R_{q_r}(Y_i, Y_j)Y_j, Y_k \rangle.$$

As shown in Section 6, q_r is a Riemannian submersion metric with base \mathbf{CH}^{n-1} and totally geodesic fiber $\frac{v}{h}\mathbf{S}^1$. If $v = \sinh(r)$ and $h = \cosh(\frac{r}{2})$, we denote q_r by q_r^{sc} . Fix an arbitrary $r > 0$, and let t be the positive number satisfying

$$\sqrt{t} \frac{\sinh(r)}{\cosh(\frac{r}{2})} = \frac{v}{h},$$

so that q_r is obtained from the Riemannian submersion metric q_r^{sc} by rescaling the fiber by t . The curvature tensors of q_r , q_r^{sc} is related by O'Neill's formulas [Bes87, Theorem 9.28ce and Lemma 9.69ac] via T and A tensors of the submersions, and as we show below q_r , q_r^{sc} have proportional $ijjk$ -components of the curvature tensor, which would finish the proof.

It remains to show proportionality of $ijjk$ -components. The Riemannian submersion metric q_r satisfies $T = 0$, as the fibers are totally geodesic, and $\langle (\nabla_{Y_1} A)_{Y_j} Y_k, Y_1 \rangle_{q_r} = 0$ for distinct $j, k > 1$, as follows e.g. from the last identity in [Bes87, 9.32]; in essence this term vanishes because the fiber is one-dimensional.

So by [Bes87, Theorem 9.28c]

$$\langle R_{q_r}(Y_i, Y_1)Y_1, Y_k \rangle_{q_r} = -\langle A_{Y_i}Y_1, A_{Y_k}Y_1 \rangle_{q_r}$$

where by [Bes87, Lemma 9.69a] the right hand side is the t^2 -multiple of the same quantity for q_{sc} , which equals to $\langle R_{q_r^{sc}}(Y_i, Y_1)Y_1, Y_k \rangle_{q_r^{sc}} = 0$.

Similarly, by [Bes87, Theorem 9.28e],

$$\langle R_{q_r}(Y_i, Y_j)Y_j, Y_1 \rangle_{q_r} = \langle (\nabla_{Y_j} A)_{Y_i} Y_j, Y_1 \rangle_{q_r}$$

where by [Bes87, Lemma 9.69c] the right hand side is the t -multiple of the same quantity for q_r^{sc} , which equals to $\langle R_{q_r^{sc}}(Y_i, Y_j)Y_j, Y_1 \rangle_{q_r^{sc}} = 0$. The case when Y_1 occupies the first slot, instead of the last, follows by the symmetry of the curvature tensor.

Finally, if $1 \notin \{i, j, k\}$, then by [Bes87, Theorem 9.28f]

$$\langle R_{q_r}(Y_i, Y_j)Y_j, Y_k \rangle_{q_r} = \frac{1}{h^4} \langle R_k(\check{X}_i, \check{X}_j)\check{X}_j, \check{X}_k \rangle_k - 3\langle A_{Y_i}Y_j, A_{Y_j}Y_k \rangle_{q_r}$$

where the first summand vanishes by (5.2), and the second summand is the t -multiple of the same quantity for q_r^{sc} equal to $\langle R_{q_r^{sc}}(Y_i, Y_j)Y_j, Y_k \rangle_{q_r^{sc}} = 0$. \square

9. SECTIONAL CURVATURE

In this section we compute the sectional curvature of $\lambda_{v,h}$ in terms of v, h . Fix an arbitrary 2-plane σ that is tangent to $I \times F$ at the point $z \in \{r\} \times F$. As in Section 4, we denote the projection $I \times F \rightarrow \mathbf{CH}^{n-1}$ by p , and let $w = p(z) \in \mathbf{CH}^{n-1}$.

We first focus on the “generic” case when the subspace $dp(\sigma) \subset T_w \mathbf{CH}^{n-1}$ is 2-dimensional, and treat the case of $\dim(dp(\sigma)) < 2$ in Remark 9.7.

To simplify the computation we choose a frame $\{Y_i\}$ depending on the position of σ . Since $\{r\} \times F \subset I \times F$ has codimension one, σ contains a unit vector D that is tangent to $\{r\} \times F$. Let $H_2 \in dp(\sigma)$ be a unit vector proportional to $dp(D)$, and let $H_3 \in dp(\sigma)$ be a vector such that $\{H_2, H_3\}$ is an orthonormal basis of $dp(\sigma)$. As in Section 4, we extend $\{H_2, H_3\}$ to the frame $\{\check{X}_2, \dots, \check{X}_{2n-1}\}$ in \mathbf{CH}^{n-1} satisfying $\check{X}_2 = H_2$, $\check{X}_3 = H_3$ at w , and then lift each \check{X}_i to a horizontal vector field X_i , so that $Y_i = X_i/h$ is the corresponding unit vector field. Thus $\partial_r, Y_1, Y_2, \dots, Y_{2n-1}$ is a local frame near z .

Since D, Y_2 are tangent to $\{r\} \times F$, and $dp(D)$ is proportional to $dp(Y_2)$, we conclude that D lies in the span of Y_1, Y_2 . Let $C \in \sigma$ be a unit vector which is orthogonal to D . By construction $dp(C) \subset dp(\sigma)$ lies in the span of $dp(Y_2), dp(Y_3)$, so C lies in the span of $\partial_r, Y_1, Y_2, Y_3$. Thus $\{C, D\}$ is an orthonormal basis in σ such that

$$C = c_0 \partial_r + c_1 Y_1 + c_2 Y_2 + c_3 Y_3, \quad D = d_1 Y_1 + d_2 Y_2,$$

for some $c_i, d_j \in \mathbb{R}$.

For brevity, in this section we suppress subscript λ in the metric and curvature tensors of λ , and also denote by K the sectional curvature of λ . Symmetries of the curvature tensor, and Section 8 imply the following.

$$\begin{aligned} K(C, D) &= d_1^2 \langle R(C, Y_1) Y_1, C \rangle + d_2^2 \langle R(C, Y_2) Y_2, C \rangle + 2d_1 d_2 \langle R(C, Y_1) Y_2, C \rangle \\ \langle R(C, Y_1) Y_1, C \rangle &= c_2^2 K(Y_2, Y_1) + c_3^2 K(Y_3, Y_1) + c_0^2 K(\partial_r, Y_1), \\ \langle R(C, Y_2) Y_2, C \rangle &= c_1^2 K(Y_2, Y_1) + c_3^2 K(Y_3, Y_2) + c_0^2 K(\partial_r, Y_2), \\ \langle R(C, Y_1) Y_2, C \rangle &= -c_1 c_2 K(Y_2, Y_1) + \frac{3}{2} c_0 c_3 \langle R(\partial_r, Y_1) Y_2, Y_3 \rangle, \end{aligned}$$

where by Section 8 all but two mixed terms vanish, and the nonzero mixed terms add up to $\frac{3}{2} c_0 c_3 \langle R(\partial_r, Y_1) Y_2, Y_3 \rangle$. Thus $K(C, D)$ equals to

$$(9.1) \quad (d_1 c_2 - d_2 c_1)^2 K(Y_2, Y_1) + d_1^2 c_3^2 K(Y_3, Y_1) + d_1^2 c_0^2 K(\partial_r, Y_1) + d_2^2 c_3^2 K(Y_3, Y_2) + d_2^2 c_0^2 K(\partial_r, Y_2) + 3d_1 d_2 c_0 c_3 \langle R(\partial_r, Y_1) Y_2, Y_3 \rangle,$$

where it follows from (7.1)–(7.3) and (8.1)–(8.2) that

$$(9.2) \quad K(Y_2, Y_1) = K(Y_3, Y_1) = \frac{v^2}{16h^4} - \frac{v'}{v} \frac{h'}{h},$$

$$(9.3) \quad K(Y_3, Y_2) = -\frac{1}{4h^2} - \frac{3}{h^2} c_{23}^2 - 3c_{23}^2 \frac{v^2}{4h^4} - \left(\frac{h'}{h} \right)^2,$$

$$(9.4) \quad K(\partial_r, Y_1) = -\frac{v''}{v}, \quad K(\partial_r, Y_2) = -\frac{h''}{h},$$

$$(9.5) \quad \langle R(\partial_r, Y_1) Y_2, Y_3 \rangle = -c_{23} \frac{v}{h^2} \left(\ln \frac{v}{h} \right)' = -c_{23} \frac{v}{h^2} \left(\frac{v'}{v} - \frac{h'}{h} \right).$$

Remark 9.6. Since C, D are orthonormal, $d_1 c_1 + d_2 c_2 = 0$ so

$$(d_1 c_2 - d_2 c_1)^2 = (d_1 c_2 - d_2 c_1)^2 + (d_1 c_1 + d_2 c_2)^2 = (d_1^2 + d_2^2)(c_1^2 + c_2^2) = c_1^2 + c_2^2.$$

In particular, if the mixed term vanishes and the sectional curvatures of coordinate planes are bounded above by a negative constant, then $K(\sigma)$ is bounded above by the same constant as the coefficients add up to 1:

$$c_1^2 + c_2^2 + d_1^2 c_3^2 + d_1^2 c_0^2 + d_2^2 c_3^2 + d_2^2 c_0^2 = c_1^2 + c_2^2 + (d_1^2 + d_2^2)(c_0^2 + c_3^2) = 1.$$

Remark 9.7. If $dp(\sigma)$ is zero-dimensional, then σ is the $Y_1 \partial_r$ -plane, whose sectional curvature is given by (9.4). If $dp(\sigma)$ is one-dimensional, then σ intersects the $Y_1 \partial_r$ -plane in a line, and we let D be a unit vector that spans the line, so $D = d_0 \partial_r + d_1 Y_1$ with $d_0, d_1 \in \mathbb{R}$. Let C be a unit vector in σ

that is orthogonal to D . Then $dp(\sigma)$ is a nonzero subspace spanned by $dp(C)$, and we let H_2 be a unit vector that is proportional to $dp(C)$. Extending H_2 to a frame $\{\check{X}_2, \dots, \check{X}_{2n-1}\}$ in \mathbf{CH}^{n-1} satisfying $\check{X}_2 = H_2$, we get a frame $\partial_r, Y_1, \dots, Y_{2n-1}$ near z in which $C = c_0\partial_r + c_1Y_1 + c_2Y_2$ with $c_0, c_1, c_2 \in \mathbb{R}$. Repeating the above arguments, we easily compute $K(C, D)$, and in fact all the mixed terms now vanish so that

$$(9.8) \quad K(C, D) = (d_0c_1 - d_1c_0)^2 K(\partial_r, Y_1) + d_0^2 c_2^2 K(\partial_r, Y_2) + d_1^2 c_2^2 K(Y_1, Y_2),$$

where again $d_0c_0 + d_1c_1 = 0$ implies that $(d_0c_1 - d_1c_0)^2 = c_1^2 + c_2^2$, so that if the sectional curvatures of coordinate planes are bounded above by a negative constant, then $K(\sigma)$ is bounded above by the same constant.

10. CONSTRUCTION OF THE METRIC AND CURVATURE ESTIMATES

In this section we construct the functions v, h such that $\sec(\lambda_{v,h}) \leq k$ for some negative number k , and $\lambda_{v,h}$ agrees with the complex hyperbolic metric, i.e. $v = \sinh(r)$, $h = \cosh(r/2)$, when r is at least half of the normal injectivity radius of S . The domain of v, h will be the interval from $-\infty$ to the normal injectivity radius of S .

Let ϵ be a small positive parameter such that 8ϵ is less than the normal injectivity radius of S in M . When precise estimates are unimportant we use the “big O ” notation, and rely on smallness of ϵ without further mention.

Defining v by bending $\sinh(r)$ to ϵe^r . Let r_ϵ be the unique solution of the equation $\sinh(r) = \epsilon e^r$; thus $-2r_\epsilon = \ln(1 - 2\epsilon)$ so that $r_\epsilon = \epsilon + O(\epsilon^2) \approx \epsilon$. Let $r_\epsilon^- := r_\epsilon - \epsilon^4$.

Proposition 10.1. *There is a C^1 function \mathbf{v} and $r_\epsilon^+ \in (r_\epsilon, r_\epsilon + \epsilon^4]$ such that*

- (1) \mathbf{v} is positive and increasing,
- (2) $\mathbf{v}(r) = \sinh(r)$ for $r \geq r_\epsilon^+$,
- (3) $\mathbf{v}(r) = \epsilon e^r$ for $r \leq r_\epsilon^-$,
- (4) if $r \in [r_\epsilon^-, r_\epsilon^+]$, then \mathbf{v} is C^∞ , $\mathbf{v}''(r) > \mathbf{v}(r)$, and $(\ln(\mathbf{v}))'' > 0$.

Proof. The slope of $\ln(\sinh(r))$ at r_ϵ is $\coth(r_\epsilon) \gg 1$, so the graphs of $\ln(\sinh(r))$ and $r + \ln(\epsilon)$ intersects transversely at r_ϵ .

Since $\ln(\sinh(r))'' = -\frac{1}{\sinh^2(r)} < 0$, the function $\ln(\sinh(r))$ is (strictly) concave, so given $r_\epsilon^+ \in (r_\epsilon, r_\epsilon + \epsilon^4)$ the tangent line l_ϵ^+ to $\ln(\sinh(r))$ at r_ϵ^+ intersects the line $l_\epsilon^-(r) = r + \ln(\epsilon)$ at $r_\epsilon^0 < r_\epsilon$. (This becomes obvious after drawing graphs of $\ln(\sinh(r))$, l_ϵ^- near r_ϵ . Alternatively, the lines l_ϵ^+ , l_ϵ^- intersect transversality, and they cannot intersect at a point $r \geq r_\epsilon$ because $r \geq r_\epsilon$ implies $l_\epsilon^+(r) \geq \ln(\sinh(r)) \geq l_\epsilon^-(r)$, where the first inequality follows from concavity of $\ln(\sinh(r))$, and the equalities occur at different points r_ϵ^+ , r_ϵ).

Since $r_\epsilon^0 \rightarrow r_\epsilon$ as $r_\epsilon^+ \rightarrow r_\epsilon$, so we may assume that $r_\epsilon^0 \in (r_\epsilon^-, r_\epsilon)$. Note that r_ϵ^0 is the only nonsmooth point of the piecewise-linear function $l(r) := \max\{l^-(r), l^+(r)\}$. The slope of l^- is 1, and the slope of l^+ is $\coth(r_\epsilon^+) > 1$, so l is convex. Restricting l to $[r_\epsilon^-, r_\epsilon^+]$, we let w_l be the smoothing of l given by Proposition A.4 for some small δ . Thus w_l is a C^∞ increasing function defined on $[r_\epsilon^-, r_\epsilon^+]$ and such that $w_l'' > 0$, and the graphs of l , w_l touch at the points $r_\epsilon^-, r_\epsilon^+$.

Let w be the function equal to $r + \ln(\epsilon)$ for $r \leq r_\epsilon^-$, equal to w_l for $r \in [r_\epsilon^-, r_\epsilon^+]$, and equal to $\ln(\sinh(r))$ for $r \geq r_\epsilon^+$. Then w is an increasing C^1 function, and the function $\mathbf{v} := e^w$ is positive, increasing, and C^1 , and furthermore, the restrictions of \mathbf{v} to $(-\infty, r_\epsilon^-]$, $[r_\epsilon^-, r_\epsilon^+]$, $[r_\epsilon^+, \infty)$ are C^∞ .

Finally, assume $r \in [r_\epsilon^-, r_\epsilon^+]$, and consider the function e^{w_l} , i.e. the restriction of \mathbf{v} to $[r_\epsilon^-, r_\epsilon^+]$. Certainly, $(\ln(\mathbf{v}))'' = w_l'' > 0$. Since $\frac{\mathbf{v}'}{\mathbf{v}} = w_l'$ is increasing, $0 < (\frac{\mathbf{v}'}{\mathbf{v}})' = \frac{\mathbf{v}''}{\mathbf{v}} - (\frac{\mathbf{v}'}{\mathbf{v}})^2$. Hence $\frac{\mathbf{v}''}{\mathbf{v}} > (\frac{\mathbf{v}'}{\mathbf{v}})^2 \geq 1$, where the last inequality holds because $\frac{\mathbf{v}'}{\mathbf{v}}$ is bounded below by its value at r_ϵ^- which equals to 1, because it can be computed using $\mathbf{v} = \epsilon e^r$. \square

Proposition 10.2. *For each small positive ϵ there exists $\delta_0 > 0$, and a C^∞ function $v = v(r)$ depending on the parameter $\delta \in (0, \delta_0)$ such that*

- v is positive and increasing,
- $v(r) = \mathbf{v}(r)$ if r is outside the ϵ^8 -neighborhood of $\{r_\epsilon^-, r_\epsilon^+\}$,
- if r is in the ϵ^8 -neighborhood of $\{r_\epsilon^-, r_\epsilon^+\}$, then $\frac{v''}{v} > 1 + O(\epsilon)$,
- if ϵ is fixed, then v converges to \mathbf{v} in uniform C^1 topology as $\delta \rightarrow 0$.

Proof. We define $v := \mathbf{v}_{\delta, \sigma}$ to be the smoothing of \mathbf{v} at $r_\epsilon^-, r_\epsilon^+$, given by Lemma A.1. In particular, v is positive and increasing, $v = \mathbf{v}$ is outside the σ -neighborhood of $\{r_\epsilon^-, r_\epsilon^+\}$, and v converges to \mathbf{v} uniformly in C^1 topology as $\delta \rightarrow 0$. If r in the σ -neighborhood of $[r_\epsilon^-, r_\epsilon^+]$, then $\mathbf{v}(r) < \mathbf{v}(r_\epsilon^+ + 2\sigma)$, so if δ is small enough, then $v(r) < \mathbf{v}(r_\epsilon^+ + 2\sigma)$.

By Proposition 10.1, if $r \in [r_\epsilon^-, r_\epsilon^+]$ then $\mathbf{v}''(r) > \mathbf{v}(r) > \mathbf{v}(r_\epsilon^- - 2\sigma)$, where by $\mathbf{v}''(r)$ at $r_\epsilon^-, r_\epsilon^+$, we mean one-sided derivatives. If $r \in [r_\epsilon^- - \sigma, r_\epsilon^+ + \sigma] \setminus (r_\epsilon^-, r_\epsilon^+)$, then \mathbf{v} equals to ϵe^r or $\sinh(r)$, so $\mathbf{v}''(r) = \mathbf{v}(r) > \mathbf{v}(r_\epsilon^- - 2\sigma)$. Therefore, Lemma A.1 implies that $v'' > \mathbf{v}(r_\epsilon^- - 2\sigma)$.

Therefore, for small δ and $\sigma = \epsilon^8$

$$\frac{v''}{v} > \frac{\mathbf{v}(r_\epsilon^- - 2\sigma)}{\mathbf{v}(r_\epsilon^- + 2\sigma)} = \frac{\epsilon e^{r_\epsilon^- - 2\sigma}}{\sinh(r_\epsilon^- + 2\sigma)} = 1 + O(\epsilon).$$

provided r lies in the ϵ^8 -neighborhood of $\{r_\epsilon^-, r_\epsilon^+\}$. \square

Defining h by bending from $\cosh(r/2)$ to $e^{r/2}$. Let $\rho_\epsilon = \frac{r_\epsilon^-}{2}$ so that $\rho_\epsilon < r_\epsilon^- = r_\epsilon - e^4 \approx \epsilon$, and $\rho_\epsilon = \frac{\epsilon}{2} + O(\epsilon^2) \approx \frac{\epsilon}{2}$. The tangent line to the graph of $\cosh(\frac{r}{2})$ at ρ_ϵ is

$$(10.3) \quad l(r) = \cosh\left(\frac{\rho_\epsilon}{2}\right) + \frac{1}{2} \sinh\left(\frac{\rho_\epsilon}{2}\right) (r - \rho_\epsilon).$$

Let $q(r) := l(r) + \epsilon^6(r - \rho_\epsilon)^2$, so that the graphs of q and l touch at ρ_ϵ .

Proposition 10.4. *There is a C^1 function \mathbf{h} and $n_\epsilon < m_\epsilon < \rho_\epsilon$ such that*

- (1) \mathbf{h} is positive and increasing,
- (2) $\mathbf{h}(r) = \cosh(\frac{r}{2})$ for $r \geq \rho_\epsilon$,
- (3) $\mathbf{h}(r) = q(r)$ for $r \in [m_\epsilon, \rho_\epsilon]$,
- (4) if $r \in [n_\epsilon, m_\epsilon]$, then \mathbf{h} is C^∞ , $\mathbf{h}''(r) > \mathbf{h}(r)/4$, and $(\ln(\mathbf{h}))'' > 0$, and $\frac{\mathbf{h}'}{\mathbf{h}} \in [\frac{1}{2}, \frac{3}{4}]$.
- (5) if $r \leq n_\epsilon$, then $\mathbf{h}(r) = e^{r/2}$.

Proof. One computes that q is increasing on $[-\frac{1}{\epsilon^2}, \rho_\epsilon]$, and $q(-\frac{1}{\epsilon^2}) < 0$ while $q(0) > 0$ so the parabola q has exactly one zero z_ϵ in $(-\frac{1}{\epsilon^2}, 0)$. Therefore, on the interval $(z_\epsilon, \rho_\epsilon]$ the slope $\frac{q'}{q}$ of the function $\ln(q)$ varies from $+\infty$ to $\frac{1}{2} \tanh(\frac{\rho_\epsilon}{2}) = O(\epsilon)$. So $(z_\epsilon, \rho_\epsilon]$ contains a point m_ϵ where $\frac{q'}{q} = \frac{3}{4}$. Let L^+ be the tangent line to the graph of $\ln(q)$ at m_ϵ , and let $L^-(r) = r/2$.

Lemma 10.5. $L^+(m_\epsilon) > L^-(m_\epsilon)$.

Proof of Lemma 10.5. Since $L^+(m_\epsilon) = \ln(q(m_\epsilon))$ and $L^-(r) = r/2$, we need to show that $q(m_\epsilon) > e^{m_\epsilon/2}$. As $q \geq l$, it suffices to show that $l(m_\epsilon) > e^{m_\epsilon/2}$. Using $q' = 3q/4$ at the point m_ϵ , and $q(r) = l(r) + \epsilon^6(r - \rho_\epsilon)^2$ we derive that $l' = 3l/4 + O(\epsilon^2)$ at m_ϵ . Denote $2l'(m_\epsilon) = \sinh(\frac{\rho_\epsilon}{2})$ by x ; note that $x = \epsilon/4 + O(\epsilon^2)$. Then

$$l(m_\epsilon) = \frac{4l'(m_\epsilon)}{3} + O(\epsilon^2) = \frac{2x}{3} + O(\epsilon^2),$$

while (10.3) implies $l(m_\epsilon) = 1 + m_\epsilon \frac{x}{2} + O(\epsilon^2)$. Thus $m_\epsilon = \frac{4}{3} - \frac{2}{x} + O(\epsilon) < 2 - \frac{2}{x}$, so $e^{m_\epsilon/2} \leq e^{1 - \frac{1}{x}}$. On the other hand, $l(m_\epsilon) = \frac{2x}{3} + O(\epsilon^2) > \frac{x}{2}$. So it remains to show that $\frac{x}{2} > e^{1 - \frac{1}{x}}$, or equivalently $xe^{\frac{1}{x}} > 2e$. One computes that $xe^{\frac{1}{x}}$ decreases if $x \in (0, 1)$ and so for $0 < x < \frac{1}{5}$, we get $xe^{\frac{1}{x}} > \frac{e^5}{5} > 2e$, and Lemma 10.5 is proved. \square

Combining Lemma 10.5 with the fact that the slope of L^+ is $\frac{q'}{q}(m_\epsilon) = \frac{3}{4}$, and the slope of L^- is $\frac{1}{2}$ implies that $L^+(r) = L^-(r)$ for some $r < m_\epsilon$, which is the only nonsmooth point of the convex piecewise-linear function $L :=$

$\max\{L^-, L^+\}$. Let us fix an arbitrary $n_\epsilon < r$. Restricting L to $[n_\epsilon, m_\epsilon]$, we let ω_L be the smoothing of L given by Proposition A.4 for some small δ . Thus ω_L is a C^∞ increasing function defined on $[n_\epsilon, m_\epsilon]$ and such that $\omega_L'' > 0$, and the graphs of L, ω_L touch at the points n_ϵ, m_ϵ .

Let ω be the function equal to $\frac{r}{2}$ for $r \leq n_\epsilon$, equal to ω_L for $r \in [n_\epsilon, m_\epsilon]$, equal to $\ln(q)$ for $r \in [m_\epsilon, \rho_\epsilon]$, and equal to $\ln(\cosh(\frac{r}{2}))$ for $r \geq \rho_\epsilon$. Then ω is an increasing C^1 function, and the function $\mathbf{h} = e^\omega$ is positive, increasing, C^1 , and furthermore the restrictions of \mathbf{h} to $(-\infty, n_\epsilon]$, $[n_\epsilon, m_\epsilon]$, $[m_\epsilon, \rho_\epsilon]$, $[\rho_\epsilon, +\infty)$ are C^∞ , and furthermore, $\mathbf{h}(r) = e^{r/2}$ if $r \leq n_\epsilon$, and $\mathbf{h} = q$ if $r \in [m_\epsilon, \rho_\epsilon]$, and $\mathbf{h} = \cosh(\frac{r}{2})$ if $r \geq \rho_\epsilon$.

Finally, assume $r \in [n_\epsilon, m_\epsilon]$, and consider the function e^{ω_L} , i.e. the restriction of \mathbf{h} to $[r_\epsilon^-, r_\epsilon^+]$. Certainly, $(\ln(\mathbf{h}))'' = \omega_L'' > 0$. Since $\frac{\mathbf{h}'}{\mathbf{h}} = \omega_L'$ is increasing, $0 < (\frac{\mathbf{h}'}{\mathbf{h}})' = \frac{\mathbf{h}''}{\mathbf{h}} - (\frac{\mathbf{h}'}{\mathbf{h}})^2$. Hence $\frac{\mathbf{h}''}{\mathbf{h}} > (\frac{\mathbf{h}'}{\mathbf{h}})^2 \geq \frac{1}{4}$, where the last inequality holds because $\frac{\mathbf{h}'}{\mathbf{h}}$ is bounded below by its value at n_ϵ which equals to $\frac{1}{2}$, because it can be computed using $\mathbf{h}(r) = e^{r/2}$. Since $\frac{\mathbf{h}'}{\mathbf{h}} = \omega_L'$ is increasing, $\frac{\mathbf{h}'}{\mathbf{h}}$ varies on $[n_\epsilon, m_\epsilon]$ between its values at endpoints, where \mathbf{h} equals to $e^{r/2}$ and q , hence $\frac{\mathbf{h}'}{\mathbf{h}} \in [\frac{1}{2}, \frac{3}{4}]$ when $r \in [n_\epsilon, m_\epsilon]$. \square

Proposition 10.6. *For each small ϵ and each $\sigma \in (0, \epsilon^8)$ there is $\delta_0 > 0$, and there exists a C^∞ function $h = h(r)$ depending on the parameters ϵ, σ , and $\delta \in (0, \delta_0)$ such that*

- h is positive and increasing,
- $h(r) = \mathbf{h}(r)$ if r is outside the σ -neighborhood of $\{n_\epsilon, m_\epsilon, \rho_\epsilon\}$,
- if r is in the σ -neighborhood of $\{m_\epsilon, \rho_\epsilon\}$, then $\frac{h''}{h} > \epsilon^6$.
- if r is in the σ -neighborhood of n_ϵ , then $\frac{h''}{h} > \frac{1}{9}$,
- if ϵ, σ are fixed, then h converges to \mathbf{h} in uniform C^1 topology as $\delta \rightarrow 0$.

Proof. Let $h := \mathbf{h}_{\delta, \sigma}$ be the smoothing of \mathbf{h} at $n_\epsilon, m_\epsilon, \rho_\epsilon$, given by Lemma A.1. In particular, h is positive and increasing, $h = \mathbf{h}$ is outside the σ -neighborhood of $\{n_\epsilon, m_\epsilon, \rho_\epsilon\}$, and h converges to \mathbf{h} uniformly in C^1 topology as $\delta \rightarrow 0$.

To establish the desired lower bounds on $\frac{h''}{h}$ we need to look at one-sided second derivatives \mathbf{h}'' and then apply Lemma A.1 to derive a lower bound on h'' . In the σ -neighborhood of ρ_ϵ the one-sided second derivatives satisfy

$$\mathbf{h}'' \geq \min\{2\epsilon^6, \frac{1}{4} \cosh(\frac{\rho_\epsilon}{2})\} = 2\epsilon^6,$$

so Lemma A.1 implies that $h'' > \frac{3\epsilon^6}{2}$ for small δ . As $h(r) < \mathbf{h}(\rho_\epsilon + 2\sigma)$ for small δ , we conclude that $\frac{h''}{h} > \frac{3\epsilon^6}{2 \cosh(\rho_\epsilon + 2\sigma)} > \epsilon^6$ where the last inequality holds if say $\sigma < \epsilon^8$.

By Proposition 10.4, if $r \in [n_\epsilon, m_\epsilon]$, then $\mathbf{h}''(r) > \frac{\mathbf{h}(r)}{4}$. So if $r \in [n_\epsilon, n_\epsilon + \sigma]$, then $\mathbf{h}''(r) > \frac{\mathbf{h}(r)}{4} > \frac{\mathbf{h}(n_\epsilon - 2\sigma)}{4}$, while if $r \in [n_\epsilon - \sigma, n_\epsilon]$, then $\mathbf{h}'' = \frac{\mathbf{h}}{4} > \frac{\mathbf{h}(n_\epsilon - 2\sigma)}{4}$. So if r is in the σ -neighborhood of n_ϵ , and δ is small, then Lemma A.1 implies $h''(r) > \frac{\mathbf{h}(n_\epsilon - 2\sigma)}{4}$ and $h(r) < \mathbf{h}(n_\epsilon + 2\sigma)$, and thus

$$\frac{h''}{h} > \frac{\mathbf{h}(n_\epsilon - 2\sigma)}{4\mathbf{h}(n_\epsilon + 2\sigma)} > \frac{1}{9}.$$

where the last inequality holds provided σ is made small while ϵ is kept fixed.

Similarly, if $r \in [m_\epsilon - \sigma, m_\epsilon]$, then $\mathbf{h}''(r) > \frac{\mathbf{h}(r)}{4} > \frac{\mathbf{h}(m_\epsilon - 2\sigma)}{4}$, while if $r \in [m_\epsilon, m_\epsilon + \sigma]$, then $\mathbf{h}''(r) = 2\epsilon^6$. As $\sigma \rightarrow 0$ we have

$$\mathbf{h}(m_\epsilon - \sigma) \rightarrow \mathbf{h}(m_\epsilon) = q(m_\epsilon) > l(m_\epsilon) = \frac{\epsilon}{6} + O(\epsilon^2).$$

So for small σ we get $\mathbf{h}''(r) > 2\epsilon^6$ and $\mathbf{h}(r) \leq q(m_\epsilon + \sigma) < 2$ on the σ -neighborhood of m_ϵ . Thus if δ is small, $\frac{h''}{h} > \epsilon^6$ on the σ -neighborhood of m_ϵ . \square

Theorem 10.7. *For any sufficiently small positive ϵ there are small positive σ , δ , and a negative constant $M_{\epsilon, \sigma, \delta}$ such that $K(\lambda_{v, h}) \leq M_{\epsilon, \sigma, \delta}$.*

Remark 10.8. More precisely, there are ranges of ϵ , σ , δ for which Theorem 10.7 holds, namely, $\epsilon \in (0, \epsilon_0)$, $\sigma \in (0, \sigma_0(\epsilon))$, and $\delta \in (0, \delta_0(\epsilon, \sigma))$, i.e. the range of σ depends on ϵ and the range on δ depends on ϵ, σ .

Proof. It is enough to give a proof for 2-planes that project isomorphically to \mathbf{CH}^{n-1} , because they form a dense subset in every tangent space. The points r_ϵ^+ , r_ϵ^- , ρ_ϵ , m_ϵ , n_ϵ divide the real line into six intervals, and we estimate the curvature on each interval separately.

Step 0. Suppose $r \geq r_\epsilon^+$. Then $\mathbf{v} = \sinh(r)$ and $h = \mathbf{h} = \cosh(\frac{r}{2})$, and v converges to \mathbf{v} in C^1 topology as $\delta \rightarrow 0$, and $\frac{v''}{v} > 1 + O(\epsilon)$. If v were equal to \mathbf{v} , then the metric would be complex hyperbolic giving $K(C, D) \leq -\frac{1}{4}$. In general, the formulas (9.2)–(9.5) immediately imply that the upper curvature bound for $K(C, D)$ converges to $-\frac{1}{4} + O(\epsilon)$, as $\delta \rightarrow 0$, so that $K(C, D) \leq -\frac{1}{5}$ for all sufficiently small ϵ , δ .

Step 1. Suppose $r \in [r_\epsilon^-, r_\epsilon^+]$. Then $h = \mathbf{h} = \cosh(\frac{r}{2})$, and \mathbf{v} is positive, increasing and $(\ln(\mathbf{v}))'' > 0$. Furthermore, v converges to \mathbf{v} in C^1 topology as $\delta \rightarrow 0$, and $\frac{v''}{v} > 1 + O(\epsilon)$.

Since \mathbf{v} is increasing, $\mathbf{v}(r) \leq v(r_\epsilon + \epsilon^4) = \sinh(r_\epsilon + \epsilon^4)$, and since \mathbf{h} is increasing, we have $h(r) \geq \cosh(\frac{r_\epsilon - \epsilon^4}{2})$ so

$$\frac{\mathbf{v}}{h^2} \leq \frac{\sinh(r_\epsilon + \epsilon^4)}{\cosh(\frac{r_\epsilon - \epsilon^4}{2})} = \epsilon + O(\epsilon^2) < 2\epsilon.$$

Also since $\frac{\mathbf{v}'}{\mathbf{v}}$ is increasing, it can be estimated at endpoints where \mathbf{v} equals to ϵe^r , $\sinh(r)$. Thus

$$1 \leq \frac{\mathbf{v}'}{\mathbf{v}} \leq \coth(r_\epsilon + \epsilon^4).$$

On the other hand, $\frac{h'}{h} = \frac{1}{2} \tanh(\frac{r}{2}) = \frac{\epsilon}{4} + O(\epsilon^2)$ is small and positive, in particular,

$$0 < \frac{\mathbf{v}'}{\mathbf{v}} - \frac{h'}{h} \leq \coth(r_\epsilon + \epsilon^4) \quad \text{and} \quad \frac{h'}{h} \frac{\mathbf{v}'}{\mathbf{v}} \geq \frac{\epsilon}{4} + O(\epsilon^2) > \frac{\epsilon}{5}.$$

Also

$$\frac{\mathbf{v}}{h^2} \left(\frac{\mathbf{v}'}{\mathbf{v}} - \frac{h'}{h} \right) \leq \frac{\sinh(r_\epsilon + \epsilon^4)}{\cosh(\frac{r_\epsilon - \epsilon^4}{2})} \coth(r_\epsilon + \epsilon^4) = 1 + O(\epsilon^2).$$

If v were equal to \mathbf{v} , as happens for $r \in [r_\epsilon^- + \epsilon^8, r_\epsilon^+ - \epsilon^8]$, then the above estimates would imply the following

$$(10.9) \quad |\langle R(\partial_r, Y_1)Y_2, Y_3 \rangle| \leq |c_{23}| + O(\epsilon^2) < |c_{23}| + \epsilon,$$

$$(10.10) \quad K(Y_2, Y_1) = K(Y_3, Y_1) = \frac{\mathbf{v}^2}{16h^4} - \frac{\mathbf{v}'}{\mathbf{v}} \frac{h'}{h} < -\frac{\epsilon}{5} < 0,$$

$$(10.11) \quad K(Y_3, Y_2) < -\frac{1}{h^2} \left(\frac{1}{4} + 3c_{23}^2 \right) < -\frac{1}{\cosh^2(r_\epsilon^+)} \left(\frac{1}{4} + 3c_{23}^2 \right) < -\left(\frac{1}{5} + 3c_{23}^2 \right),$$

and since the inequalities are strict, they hold for v in place of \mathbf{v} provided δ is made small while ϵ is kept fixed, so from now on we switch to v . Since $\frac{v''}{v} > 1 + O(\epsilon)$ and $h = \cosh(\frac{r}{2})$, we get

$$K(\partial_r, Y_1) < -1 - O(\epsilon) \quad \text{and} \quad K(\partial_r, Y_2) = -\frac{1}{4}.$$

From the formula (9.1) we get

$$(10.12) \quad \begin{aligned} K(C, D) \leq m(C, D, \epsilon) := & -\frac{\epsilon}{5} \left((d_1 c_2 - d_2 c_1)^2 + d_1^2 c_3^2 \right) - \\ & (1 + O(\epsilon)) d_1^2 c_0^2 - d_2^2 c_3^2 \left(\frac{1}{5} + 3c_{23}^2 \right) - \frac{1}{4} d_2^2 c_0^2 + 3(|c_{23}| + \epsilon) |d_1 d_2 c_0 c_3| = \\ & -\frac{\epsilon}{5} \left((d_1 c_2 - d_2 c_1)^2 + d_1^2 c_3^2 \right) - \left(|d_1 c_0| \sqrt{1 + O(\epsilon)} - |d_2 c_3| \sqrt{\frac{1}{5} + 3c_{23}^2} \right)^2 + \\ & -\frac{1}{4} d_2^2 c_0^2 + |d_1 d_2 c_0 c_3| \left(3|c_{23}| + 3\epsilon - 2\sqrt{1 + O(\epsilon)} \sqrt{\frac{1}{5} + 3c_{23}^2} \right), \end{aligned}$$

where $3|c_{23}| < 2\sqrt{1 + O(\epsilon)}\sqrt{\frac{1}{5} + 3c_{23}^2}$ so every summand in $m(C, D, \epsilon)$ is nonpositive for small ϵ . In fact, if ϵ is sufficiently small and positive, then $m(C, D, \epsilon) < 0$. (Otherwise, every summand has to vanish. In particular, $|d_1 c_0| = |d_2 c_3| \sqrt{\frac{1}{5} + 3c_{23}^2}$, and $d_2 c_0 = 0$. The equation $d_2 c_0 = 0$ means that either d_2 or c_0 vanishes. If $d_2 = 0$, then $|d_1| = 1$ and $c_0 = 0$, so that $(d_1 c_2 - d_2 c_1)^2 + d_1^2 c_3^2 = c_1^2 + c_2^2 + c_3^2 = 1$. If $d_2 \neq 0$, then $c_0 = 0$, and hence $c_3 = 0$, so $(d_1 c_2 - d_2 c_1)^2 + d_1^2 c_3^2 = c_1^2 + c_2^2 = 1$. So in either case we get a contradiction with the fact that every summand vanishes).

Let $M_1(\epsilon)$ be the maximum of $m(C, D, \epsilon)$ over all orthonormal C, D ; by compactness the maximum is attained, i.e. $M_1(\epsilon) = m(C^*, D^*, \epsilon)$ for some C^*, D^* , and by the previous paragraph, $m(C^*, D^*, \epsilon) < 0$, so $K(C, D) \leq M_1(\epsilon) < 0$ for all C, D and all small positive ϵ .

Step 2. Suppose $r \in [\rho_\epsilon, r_\epsilon^-]$. If r is not in the σ -neighborhood of $\{\rho_\epsilon, r_\epsilon^-\}$, then $v(r) = \epsilon e^r$ and $h(r) = \cosh(\frac{r}{2})$, and in general v, h converge to ϵe^r , $\cosh(\frac{r}{2})$ in C^1 -topology as $\delta \rightarrow 0$, and furthermore by Propositions 10.2, 10.6 $\frac{v''}{v} \geq 1 + O(\epsilon) > \frac{1}{4}$ and $\frac{h''}{h} > \epsilon^6$. Then one computes that $\frac{v}{h^2} < 2\epsilon$ and $\frac{h'}{h} > \frac{\epsilon}{9}$ for small ϵ, δ . The formulas (9.2)–(9.5) give the following.

$$K(Y_2, Y_1) = K(Y_3, Y_1) < -\frac{\epsilon}{10},$$

$$K(Y_3, Y_2) = -\frac{1}{4h^2} < -\frac{1}{9},$$

$$K(\partial_r, Y_1) < -\frac{1}{4}, \quad K(\partial_r, Y_2) < -\epsilon^6,$$

$$|\langle R(\partial_r, Y_1)Y_2, Y_3 \rangle| = |c_{23}|(2\epsilon + O(\epsilon^2)) \leq \epsilon + O(\epsilon^2) < 2\epsilon.$$

Thus $K(C, D)$ is bounded above by

$$\begin{aligned} & -\frac{\epsilon}{10} ((d_1 c_2 - d_2 c_1)^2 + d_1^2 c_3^2) - \frac{1}{4} d_1^2 c_0^2 - \epsilon^6 d_2^2 c_0^2 - \frac{1}{9} d_2^2 c_3^2 + 6\epsilon |d_1 d_2 c_0 c_3| = \\ & -\frac{\epsilon}{10} ((d_1 c_2 - d_2 c_1)^2 + d_1^2 c_3^2) - \epsilon^6 d_2^2 c_0^2 - (\frac{1}{2} |d_1 c_0| - \frac{1}{3} |d_2 c_3|)^2 + |d_1 d_2 c_0 c_3| (6\epsilon - \frac{1}{3}), \end{aligned}$$

in which every summand is nonpositive. Then the argument of Step 1 gives a function $M_2(\epsilon)$ such that $K(C, D) \leq M_2(\epsilon) < 0$ for all C, D and all small positive ϵ .

Step 3. Suppose $r \in [m_\epsilon, \rho_\epsilon]$ so that $v(r) = \mathbf{v} = \epsilon e^r$, $\mathbf{h} = q$, and $\frac{h''}{h} > \epsilon^6$. If r is outside the σ -neighborhood of $\{m_\epsilon, \rho_\epsilon\}$ then $h = q$, and in general h converges to q in C^1 -topology as $\delta \rightarrow 0$.

On the interval $[m_\epsilon, \rho_\epsilon]$ one computes that $q' = \frac{\epsilon}{8} + O(\epsilon^2)$ and hence $q' > 0$, so that $q(r) < q(\rho_\epsilon) = \cosh(\frac{\rho_\epsilon}{2}) = 1 + O(\epsilon^2)$, while $q'' = 2\epsilon^6$, and therefore $\left(\frac{q'}{q}\right)' = \frac{qq'' - (q')^2}{q^2} < 0$, i.e. $\frac{q'}{q}$ decreases on $[m_\epsilon, \rho_\epsilon]$ from $\frac{3}{4}$ to $\frac{1}{2} \tanh(\frac{\rho_\epsilon}{2}) = \frac{\epsilon}{8} + O(\epsilon^3)$, the values of $\frac{q'}{q}$ at the endpoints of $[m_\epsilon, \rho_\epsilon]$. Thus if δ is small, then $\frac{h'}{h} \in (\frac{\epsilon}{9}, \frac{4}{5})$ on $[m_\epsilon, \rho_\epsilon]$.

Furthermore, one computes that $\frac{v}{h^2} = \frac{\epsilon e^r}{q^2}$ satisfies

$$\left(\frac{\epsilon e^r}{q^2}\right)' = 2\frac{\epsilon e^r}{q^2} \left(\frac{1}{2} - \frac{q'}{q}\right) \quad \text{and} \quad \left(\frac{\epsilon e^r}{q^2}\right)'' = 2\frac{\epsilon e^r}{q^2} \left(2\left(\frac{1}{2} - \frac{q'}{q}\right)^2 - \left(\frac{q'}{q}\right)'\right) > 0.$$

So the point where $\frac{q'}{q} = \frac{1}{2}$ is the global minimum of $\frac{e^r}{q^2}$, and the maximum is attained at the endpoints. We conclude that

$$\frac{v}{h^2} \leq \epsilon \cdot \max \left\{ \frac{e^{m_\epsilon}}{q(m_\epsilon)^2}, \frac{e^{\rho_\epsilon}}{\cosh^2(\frac{\rho_\epsilon}{2})} \right\} = \epsilon \frac{e^{\rho_\epsilon}}{\cosh^2(\frac{\rho_\epsilon}{2})} < 2\epsilon.$$

where the equality in the middle holds because $e^{m_\epsilon} < q(m_\epsilon)^2$ by Lemma 10.5. Hence for small δ we have $\frac{v}{h^2} < 2\epsilon$. In summary, for small ϵ, δ the above estimates combined with formulas (9.2)–(9.5) imply the following.

$$\begin{aligned} K(Y_2, Y_1) &= K(Y_3, Y_1) \leq \frac{\epsilon^2}{4} - \frac{\epsilon}{9} < -\frac{\epsilon}{10}, \\ K(Y_3, Y_2) &< -\frac{1}{4h^2} < -\frac{1}{4\cosh^2(\rho_\epsilon)} < -\frac{1}{9}, \\ K(\partial_r, Y_1) &= -1, \quad K(\partial_r, Y_2) \leq -\epsilon^6, \\ |\langle R(\partial_r, Y_1)Y_2, Y_3 \rangle| &\leq |c_{23}|2\epsilon(1 - \frac{\epsilon}{9}) < \epsilon. \end{aligned}$$

From the formula (9.1) we conclude that $K(C, D)$ is bounded above by

$$\begin{aligned} &-\frac{\epsilon}{10} ((d_1c_2 - d_2c_1)^2 + d_1^2c_3^2) - d_1^2c_0^2 - \epsilon^6d_2^2c_0^2 - \frac{1}{9}d_2^2c_3^2 + 3\epsilon|d_1d_2c_0c_3| = \\ &-\frac{\epsilon}{10} ((d_1c_2 - d_2c_1)^2 + d_1^2c_3^2) - \epsilon^6d_2^2c_0^2 - (|d_1c_0| - \frac{1}{3}|d_2c_3|)^2 + |d_1d_2c_0c_3|(3\epsilon - \frac{1}{3}), \end{aligned}$$

in which every summand is nonpositive. Then the argument as in Step 1 gives a function $M_3(\epsilon)$ such that $K(C, D) \leq M_3(\epsilon) < 0$ for all C, D and all small positive ϵ .

Step 4. Suppose $r \in [n_\epsilon, m_\epsilon]$ so that $v(r) = \epsilon e^r$ and $\frac{h'}{h}$ is increasing, and h converges to \mathbf{h} in C^1 -topology as $\delta \rightarrow 0$, and furthermore, $\frac{h''}{h} > \epsilon^6$.

The values of $\frac{h'}{h}$ at endpoints n_ϵ, m_ϵ are $\frac{1}{2}, \frac{3}{4}$, respectively. So $\frac{v'}{v} - \frac{h'}{h} \leq \frac{1}{2}$. Hence $\frac{v'}{v} - \frac{h'}{h} < 1$ and $\frac{h'}{h} > \frac{1}{3}$ for small δ .

Since $\ln(\mathbf{h})$ is convex, the graph of $\ln(\mathbf{h})$ is above the tangent line at n_ϵ , i.e. $\ln(\mathbf{h}(r)) \geq r/2$, so that $\mathbf{h}(r) \geq e^{r/2}$. It follows that $\frac{v}{h^2} \leq \frac{\epsilon e^r}{e^r} = \epsilon$ so that $\frac{v}{h^2} < 2\epsilon$ for small δ . The above estimates combined with formulas (9.2)–(9.5) imply the following.

$$\begin{aligned} K(Y_2, Y_1) &= K(Y_3, Y_1) < \frac{\epsilon^2}{4} - \frac{1}{3} < -\frac{1}{4}, \\ K(Y_3, Y_2) &< -\left(\frac{h'}{h}\right)^2 \leq -\frac{1}{9}, \\ K(\partial_r, Y_1) &= -1, \quad K(\partial_r, Y_2) < -\epsilon^6, \\ |\langle R(\partial_r, Y_1)Y_2, Y_3 \rangle| &\leq |c_{23}|2\epsilon \leq \epsilon. \end{aligned}$$

From the formula (9.1) we conclude that $K(C, D)$ is bounded above by

$$\begin{aligned} &-\frac{1}{4} ((d_1c_2 - d_2c_1)^2 + d_1^2c_3^2) - d_1^2c_0^2 - \epsilon^6d_2^2c_0^2 - \frac{1}{9}d_2^2c_3^2 + 3\epsilon|d_1d_2c_0c_3| = \\ &-\frac{1}{4} ((d_1c_2 - d_2c_1)^2 + d_1^2c_3^2) - \epsilon^6d_2^2c_0^2 - (|d_1c_0| - \frac{1}{3}|d_2c_3|)^2 + |d_1d_2c_0c_3|(3\epsilon - \frac{1}{3}), \end{aligned}$$

in which every summand is nonpositive. Then the argument as in Step 1 gives a function $M_4(\epsilon)$ such that $K(C, D) \leq M_4(\epsilon) < 0$ for all C, D and all small positive ϵ .

Step 5. Suppose $r \leq n_\epsilon$ so that $v(r) = \epsilon e^r$ and $\mathbf{h}(r) = e^{r/2}$ and h converges to \mathbf{h} in C^1 -topology as $\delta \rightarrow 0$, and furthermore, $\frac{h''}{h} > \frac{1}{9}$. Hence $\frac{h'}{\mathbf{h}} = \frac{1}{2}$ and $\frac{v}{\mathbf{h}^2} = \epsilon$ implying $\frac{h'}{\mathbf{h}} > \frac{1}{3}$ and $\frac{v}{\mathbf{h}^2} < 2\epsilon$.

Plugging into formulas (9.2)–(9.5) we get the following.

$$\begin{aligned} K(Y_2, Y_1) &= K(Y_3, Y_1) < \frac{\epsilon^2}{4} - \frac{1}{3} < -\frac{1}{4}, \\ K(Y_3, Y_2) &< -\left(\frac{h'}{\mathbf{h}}\right)^2 \leq -\frac{1}{9}, \\ K(\partial_r, Y_1) &= -1, \quad K(\partial_r, Y_2) < -\frac{1}{9}, \\ |\langle R(\partial_r, Y_1)Y_2, Y_3 \rangle| &\leq |c_{23}|2\epsilon(1 - \frac{1}{3}) < \epsilon. \end{aligned}$$

From the formula (9.1) we conclude that $K(C, D)$ is bounded above by

$$-\frac{1}{4}((d_1c_2 - d_2c_1)^2 + d_1^2c_3^2) - d_1^2c_0^2 - \frac{1}{9}d_2^2c_0^2 - \frac{1}{9}d_2^2c_3^2 + 3\epsilon|d_1d_2c_0c_3|$$

which is bounded above by $-\frac{1}{9} + 3\epsilon < -\frac{1}{10}$ because $|d_1d_2c_0c_3| \leq 1$ and

$$(10.13) \quad (d_1c_2 - d_2c_1)^2 + d_1^2c_3^2 + d_1^2c_0^2 + d_2^2c_3^2 + d_2^2c_0^2 = 1,$$

which completes the proof. \square

11. A-REGULAR METRICS OF NEGATIVE CURVATURE

Let $\tau_\epsilon := \epsilon e^{n_\epsilon}$; note that $0 < \tau_\epsilon < 2\epsilon$ because $n_\epsilon < \rho_\epsilon < \epsilon$. Therefore, the parameters $o_\epsilon := \ln(\tau_\epsilon)$ and $p_\epsilon := 2\ln(\tau_\epsilon)$ are negative, and go to $-\infty$ as $\epsilon \rightarrow 0$, and moreover, $p_\epsilon < o_\epsilon = \ln(\epsilon) + n_\epsilon \ll n_\epsilon$. Let $F(r) := \frac{1}{2} \frac{e^{r/2}}{\tau_\epsilon + e^{r/2}}$; this is the derivative of $\ln(\tau_\epsilon + e^{r/2})$. Note that $F' > 0$, $F \in (0, \frac{1}{2})$, and $F(p_\epsilon) = \frac{1}{4}$.

Proposition 11.1. *For each small $\epsilon > 0$ there is a C^1 function \mathbf{g} such that*

- \mathbf{g} is positive and increasing,
- if $r \geq o_\epsilon$, then \mathbf{g} coincides with the function h of Proposition 10.6, and in particular, $\mathbf{g}(r) = e^{r/2}$ for $r \in [o_\epsilon, o_\epsilon + 1]$,
- $\mathbf{g}(r) = \tau_\epsilon + e^{r/2}$ for $r \in (-\infty, p_\epsilon]$,
- if $r \in [p_\epsilon, o_\epsilon]$, then \mathbf{g} is C^∞ , and $\frac{\mathbf{g}'}{\mathbf{g}}$ is increasing, and $\frac{\mathbf{g}'}{\mathbf{g}} \in [\frac{1}{4}, \frac{1}{2}]$, and $\frac{\mathbf{g}''}{\mathbf{g}} > \left(\frac{\mathbf{g}'}{\mathbf{g}}\right)^2 \geq \frac{1}{16}$,

Proof. The function \mathbf{g} is defined outside of (p_ϵ, o_ϵ) so we just need to interpolate in between. Since $\ln(\mathbf{g})$ equals to $r/2$ on $[o_\epsilon, o_\epsilon + 1]$, it coincides with its tangent line $l^+(r) = r/2$ at o_ϵ . Let $l^-(r) = \ln(2\tau_\epsilon) + \frac{1}{4}(r - 2\ln(\tau_\epsilon))$, i.e. l^- is the tangent line to the graph of $\ln(\tau_\epsilon + e^{r/2})$ at the point $p_\epsilon = 2\ln(\tau_\epsilon)$. Then

$$l^-(p_\epsilon) = l^-(2\ln(\tau_\epsilon)) = \ln(2\tau_\epsilon) > \ln(\tau_\epsilon) = l^+(p_\epsilon).$$

On the other hand,

$$l^-(o_\epsilon) = l^-(\ln(\tau_\epsilon)) = \ln 2 + \frac{3}{4} \ln(\tau_\epsilon) < \frac{1}{2} \ln(\tau_\epsilon) = l^+(o_\epsilon),$$

hence the lines l^-, l^+ intersect on the interval (p_ϵ, o_ϵ) . The slope of l^- is $\frac{1}{4}$ which is smaller than the slope of l^+ , thus the function $l := \max\{l^-, l^+\}$ is convex and increasing. Restricting l to $[p_\epsilon, o_\epsilon]$, we let w_l be the smoothing of l given by Proposition A.4 for some small δ . Thus w_l is a C^∞ increasing function defined on $[p_\epsilon, o_\epsilon]$ and such that $w_l'' > 0$, and the graphs of l, w_l touch at the points p_ϵ, o_ϵ .

Let w be the function equal to $\ln(\tau_\epsilon + e^{r/2})$ for $r \leq p_\epsilon$, equal to w_l for $r \in [p_\epsilon, o_\epsilon]$, and equal to $\ln(h)$ for $r \geq o_\epsilon$, where h is the function of Proposition 10.6. Then w is an increasing C^1 function, and the function $\mathbf{g} := e^w$ is positive, increasing, and C^1 , and furthermore, the restrictions of \mathbf{g} to $(-\infty, p_\epsilon]$, $[p_\epsilon, o_\epsilon]$, $[o_\epsilon, \infty)$ are C^∞ .

Finally, assume $r \in [p_\epsilon, o_\epsilon]$, and consider the function e^{w_l} , i.e. the restriction of \mathbf{g} to $[p_\epsilon, o_\epsilon]$. Certainly, $(\ln(\mathbf{g}))'' = w_l'' > 0$, in other words, $\frac{\mathbf{g}'}{\mathbf{g}} = w_l'$ is increasing, hence it can be estimated at the endpoints p_ϵ, o_ϵ where \mathbf{g} equals to $\tau_\epsilon + e^{r/2}, e^{r/2}$ so that the slopes of $\frac{\mathbf{g}'}{\mathbf{g}}$ at p_ϵ, o_ϵ are $\frac{1}{4}, \frac{1}{2}$, respectively. Also $0 < (\frac{\mathbf{g}'}{\mathbf{g}})' = \frac{\mathbf{g}''}{\mathbf{g}} - (\frac{\mathbf{g}'}{\mathbf{g}})^2$. Hence $\frac{\mathbf{g}''}{\mathbf{g}} > (\frac{\mathbf{g}'}{\mathbf{g}})^2 \geq F(p_\epsilon)^2 = \frac{1}{16}$. \square

Proposition 11.2. *For each small $\epsilon > 0$ and each $\sigma \in (0, \epsilon^8)$ there is $\delta_0 > 0$, and a C^∞ function $g = g(r)$ depending on parameters ϵ, σ , and $\delta \in (0, \delta_0)$ such that*

- g is positive and increasing,
- $g(r) = \mathbf{g}(r)$ if r is outside the σ -neighborhood of $\{p_\epsilon, o_\epsilon\}$,
- if r is in the σ -neighborhood of $[p_\epsilon, o_\epsilon]$, then $\frac{g'}{g} > \frac{1}{25}$,
- if ϵ, σ are fixed, then g converges to \mathbf{g} in uniform C^1 topology as $\delta \rightarrow 0$.

Proof. We let $g := \mathbf{g}_{\delta, \sigma}$ be the smoothing of \mathbf{g} at p_ϵ, o_ϵ , given by Lemma A.1. In particular, g is positive and increasing, $g = \mathbf{g}$ is outside the σ -neighborhood of $\{p_\epsilon, o_\epsilon\}$, and g converges to \mathbf{g} uniformly in C^1 topology as $\delta \rightarrow 0$. Suppose $r \in [p_\epsilon - \sigma, p_\epsilon]$. Since $\left(\frac{\mathbf{g}'}{\mathbf{g}}\right)' = F' > 0$, we get

$$\frac{\mathbf{g}''}{\mathbf{g}} > \left(\frac{\mathbf{g}'}{\mathbf{g}}\right)^2 = F^2 > F^2(p_\epsilon - 2\sigma) > \frac{1}{25}$$

for small σ . By Proposition 11.1 the same lower bound holds on $[p_\epsilon, o_\epsilon]$, i.e. $\frac{\mathbf{g}''}{\mathbf{g}} > F^2(p_\epsilon - 2\sigma) > \frac{1}{25}$ for small σ . Finally, if $r \in [o_\epsilon, o_\epsilon + \sigma]$, then $\frac{\mathbf{g}''}{\mathbf{g}} = \frac{1}{4} >$

$F^2(p_\epsilon - 2\sigma)$ for small σ . Thus by Lemma A.1 we have $\frac{g''}{g} > \frac{1}{25}$ all small σ, δ , and r in the σ -neighborhood of $[p_\epsilon, o_\epsilon]$. \square

Theorem 11.3. *For any sufficiently small positive ϵ the metric $\lambda_{v,g}$ is A -regular, and there are positive σ, δ such that $\sec(\lambda_{v,g}) < 0$.*

Proof. Since the metric $\lambda_{v,g}$ is smooth, it is A -regular on any compact subset, hence we can assume that $r \leq p_n - \sigma$ so that $v = \epsilon e^r$ and $g = \tau_\epsilon + e^{r/2}$.

Denote $Y_0 := \partial_r$. Arguing by induction on k , we shall show that for each integer $k \in [0, \infty)$ the components of $\nabla^k R$ in the frame $\{Y_0, Y_1, \dots, Y_{2n-1}\}$ are bounded functions of r that have bounded derivatives by r .

Assume first that $k = 0$. The components of $(4, 0)$ -curvature tensor R are sums of sectional curvatures [Jos02, Lemma 3.3.3], while by (9.1), (9.8) the sectional curvature of any plane is a linear combination with constant coefficients of terms in (9.2)–(9.5). Furthermore, and this is really the key point, the terms in (9.2)–(9.5) as well as their derivatives by r are obtained from the bounded (!) functions $\frac{1}{g}$ and F by taking products, sums, and multiplying by real numbers; indeed we have:

$$\begin{aligned} \frac{v'}{v} &= 1 = \frac{v''}{v} \quad \text{and} \quad \frac{g'}{g} = F \quad \text{and} \quad F' = \frac{F}{2} - F^2 \\ \frac{g''}{g} &= \frac{F}{2} \quad \text{and} \quad \frac{v}{g^2} = 4\epsilon F^2 \quad \text{and} \quad \left(\frac{1}{g^2}\right)' = -2\frac{F}{g^2}. \end{aligned}$$

It follows that the components of R and their derivatives are linear combinations of terms what are products of functions $\frac{1}{g}$ and F , and hence are constant on any r -tube and bounded in r .

For the induction step, we fix k and let $S := \nabla^k R$. The components of the tensor ∇S are

$$(11.4) \quad Y_{i_0}(S(Y_{i_1}, \dots, Y_{i_l})) - \sum_{k=1}^l S(Y_{i_1}, \dots, Y_{i_{k+1}}, \nabla_{Y_{i_0}} Y_{i_k}, Y_{i_{k+1}}, \dots, Y_{i_l})$$

As discussed in [Bel, Appendix C], it follows from [BW04, Section 6] that

$$\nabla_{\partial_r} \partial_r = 0 = \nabla_{\partial_r} Y_k \quad \text{for } k \geq 1,$$

$$\nabla_{Y_i} \partial_r = \frac{g'}{g} Y_i = F Y_i \quad \text{for } i > 1,$$

$$\nabla_{Y_1} \partial_r = \frac{v'}{v} Y_1 = Y_1$$

By Section 4 one has $[Y_i, Y_j] = c_{ij} \frac{v}{h^2} Y_1$ for $i, j > 1$, and $[Y_i, Y_1] = 0$ (at the point z where we compute the curvature). Plugging this into Koszul's formula [Bel, Appendix C] we compute $\nabla_{Y_k} Y_l$ as follows:

$$\nabla_{Y_1} Y_1 = -\frac{v'}{v} \partial_r = -\partial_r, \quad \text{and} \quad \nabla_{Y_i} Y_i = -\frac{g'}{g} \partial_r \quad \text{if } i > 1,$$

$$\nabla_{Y_i} Y_j = c_{ij} \frac{v}{2g^2} Y_1 \quad \text{if } i, j > 1 \text{ are distinct.}$$

So if $Y_{i_0} = \partial_r$, then the induction hypothesis implies that the component (11.4) is bounded, being a linear combination with bounded coefficients of terms $S(Y_{i_1}, \dots, Y_{i_l})$ or their derivatives.

If $Y_{i_0} \neq \partial_r$, then by induction hypothesis $S(Y_{i_1}, \dots, Y_{i_l})$ is constant on r -tubes, so $Y_{i_0}(S(Y_{i_1}, \dots, Y_{i_l})) = 0$, and again the remaining terms are bounded by the induction hypothesis.

Thus the metric $\lambda_{v,g}$ is A -regular. Next we show that $\sec(\lambda_{v,g}) < 0$ following the pattern of the proof of Theorem 10.7. We only consider the generic case with the curvature given by (9.1); the non-generic case is even easier because the mixed term is not present in (9.8).

Step 1. Suppose $r \in [o_\epsilon, o_\epsilon + \sigma]$ so that $v(r) = \epsilon e^r$ and $\mathbf{g}(r) = e^{r/2}$ and g converges to \mathbf{g} in C^1 -topology as $\delta \rightarrow 0$, and furthermore, $\frac{g''}{g} > \frac{1}{25}$ for small σ . Hence $\frac{g'}{g} = \frac{1}{2}$ and $\frac{v}{g^2} = \epsilon$ implying $\frac{g'}{g} > \frac{1}{3}$ and $\frac{v}{g^2} < 2\epsilon$.

Plugging into formulas (9.2)–(9.5) we get the following:

$$\begin{aligned} K(Y_2, Y_1) &= K(Y_3, Y_1) < \frac{\epsilon^2}{4} - \frac{1}{3} < -\frac{1}{4}, \\ K(Y_3, Y_2) &< -\left(\frac{g'}{g}\right)^2 \leq -\frac{1}{9}, \\ K(\partial_r, Y_1) &= -1, \quad K(\partial_r, Y_2) < -\frac{1}{25}, \\ |\langle R(\partial_r, Y_1)Y_2, Y_3 \rangle| &\leq |c_{23}|2\epsilon(1 - \frac{1}{3}) < \epsilon, \end{aligned}$$

and we finish as in Step 5 of Theorem 10.7.

Step 2. Suppose $r \in [p_\epsilon - \sigma, o_\epsilon]$ so that $v(r) = \epsilon e^r$ and $\frac{g'}{g}$ is increasing, and g converges to \mathbf{g} in C^1 -topology as $\delta \rightarrow 0$, and also $\frac{g''}{g} > \frac{1}{25}$ for small δ . As $\frac{g'}{g} > F(p_n - \sigma) > \frac{1}{5}$ for small σ , we get $\frac{g'}{g} > \frac{1}{5}$ and $\frac{v'}{v} - \frac{g'}{g} < \frac{4}{5}$ for small δ, σ . Since $(\ln(\mathbf{g}))'' > 0$ on $[-\infty, p_\epsilon]$ and $[p_\epsilon, o_\epsilon]$, Lemma A.2, implies that $\ln(\mathbf{g})$ is strictly convex on $[p_\epsilon - \sigma, o_\epsilon]$, hence the graph of $\ln(\mathbf{g})$ is above the tangent line at o_ϵ , i.e. $\ln(\mathbf{g}(r)) \geq r/2$, so that $\mathbf{g}(r) \geq e^{r/2}$. It follows that $\frac{v}{g^2} \leq \frac{\epsilon e^r}{e^r} = \epsilon$ so that $\frac{v}{g^2} < 2\epsilon$ for small δ . The above estimates combined with formulas (9.2)–(9.5) imply the following:

$$\begin{aligned} K(Y_2, Y_1) &= K(Y_3, Y_1) < \frac{\epsilon^2}{4} - \frac{1}{5} < -\frac{1}{6}, \\ K(Y_3, Y_2) &< -\left(\frac{g'}{g}\right)^2 < -\frac{1}{25}, \\ K(\partial_r, Y_1) &= -1, \quad K(\partial_r, Y_2) < -\frac{1}{25}, \\ |\langle R(\partial_r, Y_1)Y_2, Y_3 \rangle| &\leq |c_{23}|2\epsilon^{\frac{4}{5}} < \epsilon, \end{aligned}$$

and we finish as in Step 5 of Theorem 10.7.

Step 3. Suppose $r \leq p_\epsilon - \sigma$ so that $v(r) = \epsilon e^r$ and $g = \tau_\epsilon + e^{r/2}$. We compute that $\frac{g'}{g} = F$, and $\frac{v}{g^2} = 4\epsilon F^2$, and $\frac{g''}{g} = F/2$, and deduce the following:

$$K(Y_2, Y_1) = K(Y_3, Y_1) = \epsilon^2 F^4 - F < F(\epsilon^2 - 1) < -\frac{F}{2},$$

$$K(Y_3, Y_2) < -\left(\frac{g'}{g}\right) = -F^2$$

$$K(\partial_r, Y_1) = -1, \quad K(\partial_r, Y_2) = -\frac{F}{2},$$

$$|\langle R(\partial_r, Y_1)Y_2, Y_3 \rangle| < |c_{23}|4\epsilon F^2 \leq 2\epsilon F^2.$$

Thus $K(C, D) + \frac{F}{2}((d_1 c_2 - d_2 c_1)^2 + d_1^2 c_3^2) + \frac{F}{2}d_2^2 c_0^2$ is bounded above by

$$\begin{aligned} & -d_1^2 c_0^2 - F^2 d_2^2 c_3^2 + 6\epsilon F^2 |d_1 d_2 c_0 c_3| = \\ & -(|d_1 c_0| - F|d_2 c_3|)^2 + |d_1 d_2 c_0 c_3| F(6\epsilon F - 2), \end{aligned}$$

in which every summand is nonpositive. Then the argument of Step 5 of Theorem 10.7 gives a function $M(\epsilon, r)$ with $K(C, D) \leq M(\epsilon, r) < 0$ for all C, D , where $M(\epsilon, r) \rightarrow 0$ as $r \rightarrow -\infty$ because e.g. $K(\partial_r, Y_2) = -\frac{F}{2} \rightarrow 0$ as $r \rightarrow -\infty$. \square

12. PROOF OF THEOREM 1.1

Let U be the intersection of $M \setminus S$ and a small tubular neighborhood of S

To prove (ii), equip each component of U with the complete negatively curved A -regular metric given by Theorem 11.3. By construction the metric extends the complex hyperbolic metric on $M \setminus U$, and it is clearly negatively curved and A -regular, because so is the complex hyperbolic metric. The metric has finite volume by [Bel, Remark 3.3].

To prove (i), equip each component of U with the complete negatively curved metric given by Theorem 10.7. By construction the metric extends the complex hyperbolic metric on $M \setminus U$, and its sectional curvature is clearly bounded above by a negative constant. The metric has finite volume by [Bel, Remark 3.3].

Furthermore, each end of $M \setminus S$ has a *cuspidal neighborhood* E which by definition means that E admits a Riemannian submersion onto $(-\infty, 0]$, and there exists a constant K such that the “holonomy” diffeomorphism h_t from the fiber over $\{0\}$ to the fiber over $\{t\}$ is K -Lipschitz for each t . (Indeed, each end of $M \setminus S$ corresponding to a cusp of M has a neighborhood with warped product metric $dr^2 + f_r$ where f_r is an almost flat metric on the cusp cross-section, and the “holonomy” diffeomorphism h_t is 1-Lipschitz by exponential convergence of geodesics. Each end of $M \setminus S$ that approaches S has a neighborhood with metric $dr^2 + v^2 d\theta^2 + h^2 \mathbf{k}^{n-1}$. In either case the r -coordinate projection is a Riemannian submersion with compact fibers, and h_t is 1-Lipschitz as v, h are increasing.) Then by [Bel, Theorem 4.1] the group $\pi_1(M \setminus S)$ is hyperbolic relative to the fundamental groups of the ends of $M \setminus S$.

13. PROOF OF THEOREM 1.4

Most of the assertions are proved verbatim as in [Bel, Theorem 1.1] with the following exceptions.

(4) The claim follows from Theorem 1.1 and Osin's Dehn Surgery Theorem [Osi07] provided all peripheral subgroups are fully residually hyperbolic, i.e. if H is peripheral, then for any finite subset $F \subset H$ there is a homomorphism of H onto a non-elementary hyperbolic group that is injective on F . Finitely virtually nilpotent subgroups are residually finite, hence fully residually hyperbolic. Thus we can assume that H maps onto a non-elementary hyperbolic group with infinite cyclic kernel. Let z generate the kernel. It suffices to check that any finite subset F is mapped injectively into $H/\langle z^n \rangle$ for *some* n , because the latter group is hyperbolic. If not, then for any n there exist distinct $s_n, s'_n \in S$ that get identified in Q_n . Since $\langle z^n \rangle$ is the kernel, s_n, s'_n we have $s'_n = s_n z^{nk_n}$ for some integer $k_n \neq 0$. But S is finite, so only finitely many elements of $\langle z^n \rangle$ are obtained this way, i.e. nk_n is a bounded sequence, which forces $k_n = 0$ for large n and gives a contradiction.

(5) A group satisfies the *Strong Tits Alternative* if any subgroup either contains a nonabelian free group or is virtually abelian. Tukia [Tuk94] proved the following Tits Alternative for relatively hyperbolic groups: a subgroup that does not contain a non-abelian free subgroup is either finite, or virtually- \mathbb{Z} , or lies in a peripheral subgroup. Thus it suffices to check the Strong Tits alternative for the peripheral subgroups. If M is compact, this is proved in [Bel, Theorem 1.1(6)], while if M is noncompact, then there exists a virtually nilpotent peripheral subgroup that is not virtually abelian.

(6) According to [Reb01] a relatively hyperbolic group is biautomatic provided its peripheral subgroups are biautomatic. Virtually central extensions of hyperbolic groups are biautomatic [NR97]. Polycyclic subgroups of a biautomatic group is virtually abelian [GS91], and in particular this applies to finitely generated nilpotent groups, so we have to assume M is compact.

(8) G is not $CAT(0)$ even when M is compact because centralizers need not virtually split (and for the same reason G does not act by semisimple isometries on a $CAT(0)$ space) [BH99, Theorem 1.1 (iv), page 439]. Indeed, consider any peripheral subgroup H that is an extension with hyperbolic quotient and infinite cyclic kernel generated by z . Since H is peripheral, the centralizer of z in $\pi_1(N)$ lies in H , and hence coincides with H . If the extension virtually splits, then the circle bundle would have the zero real first Chern class because if it were nonzero it would not vanish in a finite cover, so this possibility is ruled out by the following.

Lemma 13.1. *If S is a compact totally geodesic complex $(n-1)$ -submanifold of a complete complex hyperbolic n -manifold M , then the first Chern class of the normal bundle ν of S in M is nontrivial in real cohomology.*

Proof. To see that the circle bundle has nonzero real first Chern class, look at the normal bundle ν of S in M and note that by Whitney sum formula $c_1(\nu)$ is the difference between first Chern classes of i^*TM and TS where $i: S \rightarrow M$ is the inclusion. But $2\pi c_1$ is represented by the Ricci form [Bes87, 2.75], which equals to $-\frac{n+1}{2}$ -multiple of the Kähler form [KN96, Remark after Theorem IX.7.5] of the complex hyperbolic metric. Since S, M are complex hyperbolic, the Kähler form of M restricts to the Kähler form of S . One then computes that $2\pi c_1(\nu)$ is represented by $-\frac{1}{2}$ -multiple of the Kähler form of S . Since S is compact, the Kähler class is nontrivial. \square

APPENDIX A. BENDING AND SMOOTHING CONVEX FUNCTIONS

This paper relies on delicate warped product constructions, and I find it worthwhile to summarize some elementary results on bending and smoothing convex functions.

To avoid confusion we note that in this paper a function f is called *strictly convex* if $f(tx + (1-t)y) < tf(x) + (1-t)f(y)$ for all $x \neq y$ and $t \in (0, 1)$. For example, if $f'' > 0$ everywhere, then f is strictly convex, while the converse is not true: near $x = 0$ the function $f(x) = x + x^4$ is increasing strictly convex, yet $f''(0) = 0$. Similarly, f is called *strictly concave* if $-f$ is strictly convex.

Lemma A.1 modifies an argument of Ghomi [Gho02] by keeping track of the first and second derivative of the smoothing.

Lemma A.1. *Suppose that $f: [a, b] \rightarrow \mathbb{R}$ is a positive continuous function such that for some $c \in (a, b)$ the restrictions of f to $[a, c]$, $[c, b]$ are C^2 with $f'' > k$, and $f'(c_-) \leq f'(c_+)$. Then*

- (1) *for each small $\delta, \sigma > 0$ there is a C^2 function $f_{\delta, \sigma}: [a, b] \rightarrow \mathbb{R}$ that coincides with f outside the σ -neighborhood of c , and satisfies $f''_{\delta, \sigma} > k$ on $[a, b]$.*
- (2) *if f is C^l with $0 \leq l \leq \infty$ near $x \in [a, b]$, then $f_{\delta, \sigma}$ is C^l near x , and $f_{\delta, \sigma}$ converges to f in the uniform C^l -topology near x as $\delta \rightarrow 0$. If f is C^∞ away from c , then $f_{\delta, \sigma}$ is C^∞ on $[a, b]$.*
- (3) *If f is increasing and δ is small enough, then $f'_{\delta, \sigma} > 0$.*

Proof. Let $\phi_\delta: \mathbb{R} \rightarrow [0, 1]$ be a smooth (bump) function with support within $(-\delta, \delta)$, and such that $\phi_\delta = 1$ on $[-\delta/2, \delta/2]$. We use the following notations:

$$\theta_\delta := \frac{\phi_\delta}{\int_{\mathbb{R}} \phi_\delta} \quad \text{and} \quad g_{\theta_\delta}(x) := \int_{\mathbb{R}} g(x-y)\theta_\delta(y)dy = \int_{\mathbb{R}} \theta_\delta(x-y)g(y)dy.$$

It is well known (see e.g. [Hir94, Theorem 2.3]) that g_{θ_δ} is C^∞ , and if g is C^m with $0 \leq m \leq \infty$, then g_{θ_δ} converges to g in the uniform C^m -topology on $[a, b]$ as $\delta \rightarrow 0$; also if $m \geq 2$, then convolution and differentiation commute, so the m th derivative of g_{θ_δ} satisfies $(g_{\theta_\delta})^{(m)} = (g^{(m)})_{\theta_\delta}$. Consider

$$f_{\delta, \sigma}(x) := f_{\theta_\delta \theta_\delta}(x) \phi_\sigma(c - x) + f(x)(1 - \phi_\sigma(c - x)),$$

where $f_{\theta_\delta \theta_\delta}$ is the convolution of f , θ_δ , and θ_δ . Thus if f is C^l with $0 \leq l \leq \infty$ near $x \in [a, b]$, then $f_{\delta, \sigma}$ is C^l near x , and $f_{\delta, \sigma}$ converges to f in the uniform C^l -topology near x as $\delta \rightarrow 0$. Note that $f_{\delta, \sigma} = f$ when $|x - c| > \sigma$, and $f_{\delta, \sigma} = f_{\theta_\delta \theta_\delta}$ when $|x - c| < \sigma/2$, in particular, if f is C^∞ away from c , then $f_{\delta, \sigma}$ is C^∞ everywhere, which proves (2).

Convolution with any nonnegative function preserves increasing functions, hence if f is increasing, then so are f_{θ_δ} , $f_{\theta_\delta \theta_\delta}$. Thus $f'_{\delta, \sigma}$ either equals to $f'_{\theta_\delta \theta_\delta} > 0$, or converges to $f' > 0$ in C^1 topology as $\delta \rightarrow 0$, hence (3) is proved.

Since $f_{\delta, \sigma}(x)$ converges to f in C^2 topology outside the $\sigma/4$ -neighborhood of c , as $\delta \rightarrow 0$, we know that $f''_\delta(x) > k$ for small δ and $|x - c| > \sigma/4$. Since $f_{\delta, \sigma} = f_{\theta_\delta \theta_\delta}$ for $|x - c| < \sigma/2$, it remains to show that $f''_{\theta_\delta \theta_\delta} > k$.

To this end, let $q(x) := f(x) - k \frac{(x-c)^2}{2}$. Since $f'' > k$ on $[a, c]$, $[c, b]$, respectively, the restrictions of q to $[a, c]$ and $[c, b]$ satisfies $q'' > 0$. Also $q'(c_-) = f'(c_-) \leq f'(c_+) = q'(c_+)$, so by Lemma A.2, q is strictly convex on $[a, b]$, hence Lemma A.3 implies that $q''_{\theta_\delta \theta_\delta} > 0$ on $[a, b]$, but

$$q''_{\theta_\delta \theta_\delta} = f''_{\theta_\delta \theta_\delta} - \left(k \frac{(x-c)^2}{2} \right)''_{\theta_\delta \theta_\delta} = f''_{\theta_\delta \theta_\delta} - k_{\theta_\delta \theta_\delta} = f''_{\theta_\delta \theta_\delta} - k,$$

proving (1). \square

Lemma A.2. *If $a_1 < c < a_2$, and if f_1, f_2 are two strictly convex C^1 functions defined on $[a_1, c]$, $[c, a_2]$ respectively such that $f_1(c) = f_2(c)$, and $f'_1(c_-) \leq f'_2(c_+)$, then the function $f: [a_1, a_2] \rightarrow \mathbb{R}$ that equals to f_1 on $[a_1, c]$, and to f_2 on $[c, a_2]$ is strictly convex.*

Proof. Take $b_1 \in [a_1, c)$, $b_2 \in (c, a_2]$, and show that the line segment $[b_1, b_2]$ lies above the graph of f . Let λ_i be the line through $f(b_i)$, $f(c)$, and L_i be the tangent line to the graph of f_i at c . Since f_1 is strictly convex, $\lambda_1 > f_1 > L_1$ on $[a_1, c)$ so the slope of λ_1 is less than the slope of L_1 , which equals to $f'_1(c_-)$. Similarly, strict convexity of f_2 implies that $\lambda_2 > f_2 > L_2$ on $(c, a_2]$, so the slope of λ_2 is greater than the slope of L_2 which equals to $f'_2(c_+)$. Since $f'_1(c_-) \leq f'_2(c_+)$, the slope of λ_1 is less than the slope of λ_2 , and hence the function $\lambda = \max\{\lambda_1, \lambda_2\}$ is strictly convex. Hence $[b_1, b_2]$ lies above the graph of λ but strict convexity of f_1, f_2 implies that $f \leq \lambda$, so $[b_1, b_2]$ lies above the graph of f . \square

Lemma A.3. *If f is strictly convex, then $f''_{\theta_\delta \theta_\tau} > 0$.*

Proof. Convolution with any nonnegative function preserves strict convexity, so f_{θ_δ} , $f_{\theta_\delta \theta_\tau}$ are strictly convex. Differentiating under the sign of integral, we get that $f''_{\theta_\delta \theta_\tau}$ is the convolution of nonnegative smooth functions f''_{θ_δ} and θ_τ . So if $f''_{\theta_\delta \theta_\delta}(x) = 0$, then $f''_{\theta_\delta}(x - y)$ must vanish wherever $\theta_\delta(y)$ is nonzero, so $f''_{\theta_\delta} = 0$ on a neighborhood of x . It follows that f_{θ_δ} is affine near x , which contradicts the strict convexity of f_{θ_δ} . \square

The following modification of Lemma A.1 is useful.

Proposition A.4. *Given real numbers k, a_1, c, a_2 with $a_1 < c < a_2$, let $f_1: [a_1, c] \rightarrow \mathbb{R}$ and $f_2: [c, a_2] \rightarrow \mathbb{R}$ be C^2 functions satisfying $f''_i \geq k$, $f_1(c) = f_2(c)$ and $f'_1(c) < f'_2(c)$. If $f: [a_1, a_2] \rightarrow \mathbb{R}$ denotes the (continuous) function satisfying $f = f_1$ on $[a_1, c]$ and $f = f_2$ on $[c, a_2]$, then for any small $\delta > 0$ there exists a C^2 function $f_\delta: [a_1, a_2] \rightarrow \mathbb{R}$ such that*

- (1) $f''_\delta > k$
- (2) $f_\delta = f$ and $f'_\delta = f'$ at the points a_1, a_2 ,
- (3) if f is increasing, then $f'_\delta > 0$
- (4) If f is C^l on $[a_1, a_2]$ for some integer $l \in [0, \infty]$, then f_δ is C^l on $[a_1, a_2]$, and f_δ converges to f in the C^l -topology on $[a_1, a_2]$ as $\delta \rightarrow 0$.

Proof. Consider the functions

$$F_{1,\delta}(r) = f_1(r) + \delta(r - a_1)^2, \quad F_{2,\delta}(r) = f_2(r) + \delta(r - a_2)^2 \frac{(c - a_1)^2}{(c - a_2)^2}$$

defined on domains of f_1, f_2 , respectively. For each i the function $F_{i,\delta}(r)$ converges to f_i in uniform C^1 topology on the domain of f_i , as $\delta \rightarrow 0$, and furthermore, $F''_{i,\delta} > k$ for small δ , and $F_{i,\delta} - f_i$ and $F'_{i,\delta} - f'_i$ vanish at a_i . Also $F_{1,\delta}(c) = F_{2,\delta}(c)$, and $F'_{1,\delta}(c) < F'_{2,\delta}(c)$ for small δ . Let F_δ be the (continuous) function satisfying $F_\delta = F_{i,\delta}$ on the domain of f_i . Applying Lemma A.1 to smooth F_δ near c , we get a function f_δ with required properties. \square

APPENDIX B. CURVATURE OF WARPED PRODUCT METRICS

In this appendix we review some formulas for the curvature tensor of a multiply-warped product metric $dr^2 + g_r$ on $I \times F$ that were worked out in [BW04, Section 6], and corrected in [Bel].

The computation in [BW04, Section 6]) works provided at each point w of F there is a basis of vector fields $\{X_i\}$ on a neighborhood $U_w \subset F$ that is g_r -orthogonal for each r . We fix one such a basis for each w . Let $h_i(r) = \sqrt{g_r(X_i, X_i)}$ so that $Y_i = X_i/h_i$ form a g_r -orthonormal basis on U_w for any $r > 0$. Since $X_i \neq 0$ and g_r is nondegenerate, $h_i > 0$

To simplify some of the formulas below we denote $g(X, Y)$ by $\langle X, Y \rangle$, denote the vector field $\frac{\partial}{\partial r}$ by ∂_r , and reserve the notation $\frac{\partial}{\partial r}T$ for the partial derivative of the function T by r .

A straightforward tedious computation (done e.g. in [BW04, Section 6]) yields the following.

$$(B.1) \quad \langle R_g(Y_i, Y_j)Y_j, Y_i \rangle = \langle R_{g_r}(Y_i, Y_j)Y_j, Y_i \rangle - \frac{h'_i h'_j}{h_i h_j},$$

$$(B.2) \quad \langle R_g(Y_i, Y_j)Y_l, Y_m \rangle = \langle R_{g_r}(Y_i, Y_j)Y_l, Y_m \rangle \quad \text{if } \{i, j\} \neq \{l, m\},$$

$$(B.3) \quad \langle R_g(Y_i, \partial_r)\partial_r, Y_i \rangle = -\frac{h''_i}{h_i}, \quad \langle R_g(Y_i, \partial_r)\partial_r, Y_j \rangle = 0 \quad \text{if } i \neq j.$$

The following mixed term is by far the most complicated and is usually the hardest to control: by [Bel, Appendix C] $2\langle R_g(\partial_r, Y_i)Y_j, Y_k \rangle$ equals to

$$\langle [Y_i, Y_j], Y_k \rangle \left(\ln \frac{h_k}{h_j} \right)' + \langle [Y_k, Y_i], Y_j \rangle \left(\ln \frac{h_j}{h_k} \right)' + \langle [Y_k, Y_j], Y_i \rangle \left(\ln \frac{h_i^2}{h_j h_k} \right)'.$$

APPENDIX C. ACKNOWLEDGEMENTS

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REFERENCES

- [ABC⁺96] J. Amorós, M. Burger, K. Corlette, D. Kotschick, and D. Toledo, *Fundamental groups of compact Kähler manifolds*, Mathematical Surveys and Monographs, vol. 44, American Mathematical Society, Providence, RI, 1996.
- [ACT02] D. Allcock, J. A. Carlson, and D. Toledo, *Orthogonal complex hyperbolic arrangements*, Symposium in Honor of C. H. Clemens (Salt Lake City, UT, 2000), Contemp. Math., vol. 312, Amer. Math. Soc., Providence, RI, 2002, pp. 1–8.
- [Bel] I. Belegradek, *Rigidity and relative hyperbolicity of real hyperbolic hyperplane complements*, arXiv:0711.2324v1.
- [Bes87] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 10, Springer-Verlag, Berlin, 1987.
- [BH99] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften, vol. 319, Springer-Verlag, Berlin, 1999.
- [Bow] B. H. Bowditch, *Relatively hyperbolic groups*, Southampton preprint, 1999, <http://www.warwick.ac.uk/masgak/preprints.html>.
- [BW04] I. Belegradek and G. Wei, *Metrics of positive Ricci curvature on bundles*, Int. Math. Res. Not. (2004), no. 57, 3079–3096.
- [FJ98] F. T. Farrell and L. E. Jones, *Rigidity for aspherical manifolds with $\pi_1 \subset \mathrm{GL}_m(\mathbf{R})$* , Asian J. Math. **2** (1998), no. 2, 215–262.
- [Gho02] M. Ghomi, *The problem of optimal smoothing for convex functions*, Proc. Amer. Math. Soc. **130** (2002), no. 8, 2255–2259 (electronic).

- [Gol99] W. M. Goldman, *Complex hyperbolic geometry*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1999, Oxford Science Publications.
- [GS91] S. M. Gersten and H. B. Short, *Rational subgroups of biautomatic groups*, Ann. of Math. (2) **134** (1991), no. 1, 125–158.
- [Hir94] Morris W. Hirsch, *Differential topology*, Graduate Texts in Mathematics, vol. 33, Springer-Verlag, New York, 1994, Corrected reprint of the 1976 original.
- [Jos02] J. Jost, *Riemannian geometry and geometric analysis*, third ed., Universitext, Springer-Verlag, Berlin, 2002.
- [Kap05] V. Kapovitch, *Curvature bounds via Ricci smoothing*, Illinois J. Math. **49** (2005), no. 1, 259–263 (electronic).
- [KN96] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. II*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1996, Reprint of the 1969 original.
- [Laf02] V. Lafforgue, *K-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes*, Invent. Math. **149** (2002), no. 1, 1–95.
- [Lub05] A. Lubotzky, *Some more non-arithmetic rigid groups*, Geometry, spectral theory, groups, and dynamics, Contemp. Math., vol. 387, Amer. Math. Soc., Providence, RI, 2005, pp. 237–244.
- [NR97] W. D. Neumann and L. Reeves, *Central extensions of word hyperbolic groups*, Ann. of Math. (2) **145** (1997), no. 1, 183–192.
- [Osi07] D. V. Osin, *Peripheral fillings of relatively hyperbolic groups*, Invent. Math. **167** (2007), no. 2, 295–326. MR MR2270456
- [Rag84] M. S. Raghunathan, *Torsion in cocompact lattices in coverings of $\text{Spin}(2, n)$* , Math. Ann. **266** (1984), no. 4, 403–419.
- [Reb01] D. Y. Rebbechi, *Algorithmic properties of relatively hyperbolic groups*, Ph.D. thesis, Rutgers Newark, 2001, arXiv:math/0302245v1.
- [Tol93] D. Toledo, *Projective varieties with non-residually finite fundamental group*, Inst. Hautes Études Sci. Publ. Math. (1993), no. 77, 103–119.
- [Tuk94] P. Tukia, *Convergence groups and Gromov's metric hyperbolic spaces*, New Zealand J. Math. **23** (1994), no. 2, 157–187.

IGOR BELEGRADEK, SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GA 30332-0160

E-mail address: `ib@math.gatech.edu`